CHAPTER 1

Arithmetic and Geometric Sequences
LEARNING OBJECTIVES

By the end of this chapter, you should be able to
✓ explain the terms sequence, arithmetic and geometric sequence;
✓ identify arithmetic and geometric sequences;
✓ calculate the terms in arithmetic and geometric sequences;
✓ calculate the sum of terms in arithmetic and geometric sequences, and
✓ apply the concepts of arithmetic and geometric sequences to some
1.1 Introduction

• A sequence or progression is a succession of terms $T_1, T_2, T_3, \ldots$, formed according to a certain fixed rule.

• A series is the indicated sum $T_1 + T_2 + T_3 + \ldots$, of the terms of a sequence.
1.2 Arithmetic sequence

• The difference between any term and the preceding term is the same throughout.
• The common difference $d = T2 - T1 = T3 - T2 = T4 - T3 = Tn - Tn - 1$. 
Some examples of arithmetic sequences

<table>
<thead>
<tr>
<th>Arithmetic sequence</th>
<th>Common difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 4, 8, 12, 16, ...</td>
<td>d = 4</td>
</tr>
<tr>
<td>(b) 7, 5, 3, 1, ...</td>
<td>d = -2</td>
</tr>
<tr>
<td>(c) −5, −9, −13, −17, ...</td>
<td>d = -4</td>
</tr>
</tbody>
</table>
1.3 Nth term and sum of first n terms of an arithmetic sequence

- If \( a = \) the first term, \( d = \) the common difference, the nth term is given by

\[ T_n = a + (n - 1)d \]

The sum of the first \( n \) terms, \( S_n \) is

\[ S_n = \frac{n}{2} [2a + (n - 1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l) \]

where \( l = \) the last term = \( a + (n - 1)d \)
Example 1

Given the arithmetic sequence: 30, 23, 16, 9, 2, ..., find the 12th term and the sum of the first 12 terms.
Here, the first term, $a = 30$ and common difference, $d = -7$

The 12th term is

$$T_{12} = a + (n - 1)d$$

$$= 30 + (12 - 1)(-7) = -47$$

The sum of the first 12 terms,

$$S_{12} = 12/2[2(30) + (12 - 1)(-7)] = 102$$
Example 2

• Find the number of terms in the following arithmetic sequence: 12, 17, 22, ..., 67. Hence, find the sum of all the terms.
Solution

Here, \( a = 12, \ d = 17 - 12 = 5 \)

Let the number of terms be \( n \).

From \( T_n = a + (n - 1)d \), we get

\[
67 = 12 + (n - 1)5
\]

\[
55 = (n - 1)5
\]

\[
n = 12
\]

Sum of all the terms ,

\[
S_{12} = \frac{n}{2} \left[ 2a + (n - 1)d \right]
\]

\[
= \frac{12}{2}[2(12) + (12 - 1)(5)] = 474
\]
Example 3

- Find the minimum number of terms that must be taken from the following sequence:

  8, 16, 24, 32, ...

so that the sum is more than 120.
Here, $a = 8$, $d = 16 - 8 = 8$
Let the minimum number of terms be $n$.
From $S_n = \frac{n}{2}[2a + (n - 1)d]$, we get
$$S_n = \frac{n}{2}[2(8) + (n - 1)8] > 120$$
$$\frac{n}{2}[8 + 8n] > 120$$
$$4n^2 + 4n - 120 > 0$$
$$n^2 + n - 30 > 0$$
$$(n + 6)(n - 5) > 0$$
$$n > 5$$ where $n$ is a positive integer
Hence, the minimum number of terms is 6.
Example 4

Assume that you borrow RM10,000 from a financial institution at a stated interest of 1.5% per month and must repay RM1,000 per month plus interest due.

(a) Write down the formula for the nth payment.
(b) What is the final payment?
Solution

Note that
1st payment = RM1,000 + RM10,000 × 1.5% = RM1,150
2nd payment = RM1,000 + RM9,000 × 1.5% = RM1,135
3rd payment = RM1,000 + RM8,000 × 1.5% = RM1,120
is an arithmetic sequence where \( a = 1,150 \), \( d = -15 \)

(a) The \( n \)th payment = \( a + (n - 1)d \)
\[ = RM1,150 + (n - 1)(-15) \]
\[ = RM1,165 - 15n \]

(b) For final payment, \( n = 10 \)
Therefore the final payment = \( RM1,165 - (15)(10) \)
\[ = RM1015. \]
Example 5

Find the first term and the common difference of an arithmetic progression if the fourth term is 33 and the tenth term is 120.
Solution

Denote the first term by \( a \) and the common difference by \( d \), then

\[
\begin{align*}
T_4 &= a + 3d = 33 \\ 
T_{10} &= a + 9d = 120
\end{align*}
\]

Equation (2) – equation (1),

\[
6d = 87
\]

\[
d = \frac{87}{6} = 14.5
\]

Substituting \( d = 14.5 \) into equation (1),

\[
a + 3(14.5) = 33
\]

\[
a = -10.5
\]

Thus, the first term is \(-10.5\) and the common difference is \(14.5\).
Example 6

Ishak starts with a monthly salary of RM1,250 for the first year and receives an annual increment of RM80. How much is his monthly salary for the nth year of service? How much will he receive monthly for his tenth year of service?
His monthly salaries for successive years are as follows.

RM1,250, RM1,330, RM1,410, RM1,490, ...

This is an arithmetic progression with the first term RM1,250 and the common difference RM80. Therefore, the nth term is

\[ T_n = a + (n - 1)d \]

\[ = RM1,250 + (n - 1)80 \]

\[ = RM1,250 + 80n - 80 \]

\[ = RM1,170 + 80n \]

Thus, for the nth year, he receives a monthly salary of

RM(80n + 1170).

For the tenth year, he receives a monthly salary of

RM[(80 \times 10) + 1170] = RM1970
Example 7

Given that 15, s, b, c, d, e, 30 form an arithmetic progression, find the values of s, b, c, d and e.
Solution

Given the first term is 15 and the seventh term is 30, we have

\[ a = 15 \text{ and } a + 6d = 30 \]

Thus, \[ 15 + 6d = 30 \]

\[ 6d = 30 - 15 \]

\[ d = 15/6 = 2.5 \]

The arithmetic progression is 15, 17.5, 20, 22.5, 25, 27.5, 30, ...

that is, \( s = 17.5, b = 20, c = 22.5, d = 25 \) and

\( e = 27.5. \)
Example 8

In a contest, all ten finalists were given cash prizes. The first winner was given RM800, the second RM740, the third RM680 and so on.

Calculate the total amount of money awarded to all the finalists.
The prizes are RM800, RM740, RM680, ...  
It is an arithmetic progression. The first term $a = 800$, common difference $d = -60$ and $n = 10$. The total amount given is  

\[ S_n = \frac{n}{2}[2a + (n - 1)d] \]
\[ S_{10} = \frac{10}{2}[2 \times 800 + (10 - 1)(-60)] \]
\[ = \text{RM}5,300 \]

The total amount is RM5,300.
1.4 Geometric sequence

• Ratio of each term to the preceding term is the same throughout.

• To test whether a sequence is a geometric sequence or not, compute the ratio

\[ r = \frac{T_n}{T_{n-1}} \text{ for } n = 2, 3, 4, \ldots \]

If \( r \) is always the same, then it is a geometric sequence.
## Examples of geometric sequences

<table>
<thead>
<tr>
<th>Geometric sequence</th>
<th>Common ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 1, 2, 4, 8, ...</td>
<td>2</td>
</tr>
<tr>
<td>(b) 3, 9, 27, 81, ...</td>
<td>3</td>
</tr>
<tr>
<td>(c) –5, 10, –20, 40, ...</td>
<td>–2</td>
</tr>
<tr>
<td>(d) 20, 10, 5, 2.5,...</td>
<td>0.5</td>
</tr>
<tr>
<td>(e) 7, 14, 28, 56, ...</td>
<td>2</td>
</tr>
</tbody>
</table>
1.5 Nth term and sum of first n terms of a geometric sequence

- If the first term of a geometric sequence is $a$ and the common ratio is $r$, then the geometric sequence can be written as
  
  $$a, ar, ar^2, ar^3, ...$$

- The $n$th term is given by
  
  $$T_n = ar^{n-1}$$
Continue

Then the sum of the first \( n \) terms is given

\[
S_n = a(r^n - 1)/(r-1) \quad \text{for } r > 1 \quad \text{or}
\]

\[
S_n = a(1 - r^n)/(1 - r) \quad \text{for } r < 1
\]
Example 9

Given the following geometric sequence: 5, 15, 45, 135, ... , find

(a) the eighth term and the tenth term,
(b) the sum of the first eight terms
Solution

(a) This is a geometric sequence with \( a = 5 \) and \( r = 3 \).

From \( T_n = ar^n - 1 \), the 8\(^{th} \) term is
\[
T_8 = 5(3)^8 - 1 = 10,935
\]
\[
T_{10} = 5(3)^{10} - 1 = 98,415
\]

(b) From \( S_n = a(r^n - 1)/(r-1) \), we get
\[
S_8 = 5(3^8 - 1)/(3-1) = 16400
\]
Example 10

Find the number of terms in the following sequence:
2, 6, 18, ..., 39366.
Calculate the sum of all the terms.
This is a geometric sequence with $a = 2$ and $r = 3$.

Let the number of terms be $n$.

From $T_n = ar^{n-1}$, we get

$39,366 = 2(3)^{n-1}$

$19,683 = (3)^{n-1}$

$\lg 19,683 = (n - 1) \lg 3$

$n = 10$.

Hence, the number of terms is 10.
Sum of all the terms

\[ S_n = \frac{a(r^n - 1)}{(r-1)}, \]

\[ S_{10} = \frac{2(3^n - 1)}{(3-1)} \]

\[ = 59,048 \]
Example 11

Find the minimum number of terms that must be taken from the sequence: 3, 12, 48, 192, ... so that the sum is more than 300.
Solution

This is a geometric sequence with $a = 3$ and $r = 4$.

Let the minimum number of terms be $n$.

From $S_n = a(r^n - 1)/(r-1)$, we get

\[3(4^n - 1)/(4-1) > 300,\]

\[4^n > 301\]

\[n \log 4 > \log 301\]

\[n > \log 301/\log 4\]

\[n > 4.117\]

\[n = 5\]

Hence, the minimum number of terms is 5.
Example 12

Maimunah saves RM1,000 in a saving account that pays 8% compounded annually. Find the amount in her account at the end of 5 years.
Solution

Note that

amount in year 0 = 1,000

amount in year 1 = 1,000(1.08)

amount in year 2 = 1,000(1.08)^2 and so on

It is a geometric sequence with a = 1,000, r = 1.08 and n = 5

Amount in the account at the end of 5 years

= 1,000(1.08)^5

= RM1,469.33
Example 13

The third term of a geometric progression is 360 and the sixth term is 1215. Find

(a) the first term,

(b) the tenth term,

(c) the sum of the first ten terms.
(a) Denote the first term by $a$ and the common ratio by $r$.

**Then**, the third term,
$$T_3 = ar^2 = 360$$  \hspace{1cm} (1)

the sixth term,
$$T_6 = ar^5 = 1215$$  \hspace{1cm} (2)

Equation (2) ÷ equation (1), we get
$$\frac{ar^5}{ar^2} = \frac{1215}{360}$$
$$r^3 = \frac{27}{8}$$
$$r = \frac{3}{2}$$

Substitute into equation (1),
$$a\left(\frac{3}{2}\right)^2 = 360$$
$$a\left(\frac{9}{4}\right) = 360$$
$$a = 160$$

Thus, the first term is **160** and the common ratio is 1.5.
(b) The 10\textsuperscript{th} term = ar\textsuperscript{9} = 160(1.5)\textsuperscript{9} \\
= 6150.94 (to 2 d.p.)

(c) The sum of the first 10 terms is

\[ S_n = a(r^n - 1)/(r-1) \]
\[ S_{10} = 160(1.5^{10} - 1)/(1.5-1), \]
\[ = 18,132.81 \]
Example 14

The sum of the first ten terms of a geometric progression is 2,046. If the common ratio is 2, find the first term and the sum of the next three terms.
Solution

Let a = the first term of the geometric progression and r = 2

\[ S_n = \frac{a(r^n - 1)}{(r-1)} \]

2046 = \( a(2^{10} - 1)/(2-1) \)

a = \( 2064/(2^{10} - 1) \)

a = 2

Sum of the next 3 terms

= \( S_{13} - S_{10} \)

= \( 2(2^{13} - 1)/(2-1) - 2046 \)

= 16,382 - 2,046

= 14,336.
SUMMARY

1. A sequence or progression is a succession of terms $T_1, T_2, T_3, \ldots$, formed according to a certain fixed rule. A series is the sum of the terms of a sequence.

   There are many types of sequences, for example, the arithmetic and the geometric sequences.

2. Arithmetic sequence: $a, a + d, a + 2d, \ldots, a + (n - 1)d$

   The $n$th term, $T_n = a + (n - 1)d$

   Sum of the first $n$ terms, $S_n = n/2 [2a + (n - 1)d]$

   or $S_n = n/2[a + l]$

   where $a =$ first term, $d =$ common difference, $l =$ last term

   To test whether a sequence is an arithmetic sequence or not, compute $d = T_n - T_{n-1}$ for $n = 2, 3, \ldots$

   If $d$ is always the same, then it is an arithmetic sequence.
3. Geometric sequence: $a, ar, ar^2, ar^3, ..., ar^{n-1}$

A geometric sequence is a sequence in which the ratio of each term to the preceding term is the same throughout. The $n^{th}$ term is given by

$$T_n = ar^{n-1}$$

Then the sum of the first $n$ terms is given

$$S_n = a(r^n - 1)/(r-1) \quad \text{for } r > 1$$

or

$$S_n = a(1 - r^n)/(1 - r) \quad \text{for } r < 1$$
where $a =$ first term, $r =$ common ratio

- To test whether a sequence is a geometric sequence or not, compute the ratio

$$r = \frac{T_n}{T_{n-1}} \text{ for } n = 2, 3, 4, \ldots.$$ 

If $r$ is always the same, then it is a geometric sequence.