CHAPTER 2

Financial Mathematics
2.1 Introduction

- The study of financial mathematics is very important and fundamental to the understanding of the economy of a country.
2.2 Simple Interest

• Definition 1: Interest is money earned when money is invested.

• Definition 2: Interest is charge incurred when a loan or credit is OBTAINED.
LEARNING OBJECTIVES

By the end of this chapter, you should be able to

- ✓ explain the concept of simple interest;
- ✓ use the simple interest formula to calculate interest, interest rate, time and dates with data provided;
- ✓ use the simple amount formula to calculate the present and future values of some investments;
- ✓ identify four concepts of exact simple interest, ordinary simple interest, exact time and approximate time;
- ✓ apply Banker’s Rule to some investments and loan problems, and
- ✓ use the concepts of equation of values to solve some investment and loan problems.
Simple interest is the interest calculated on the original principal for the entire period it is borrowed or invested. Simple interest formula

\[ I = Prt \]

where \( I \) = simple interest

\( P \) = principal

\( r \) = rate of simple interest

\( t \) = time or term in years
Example 1

RM1,000 is invested for two years in a bank, earning a simple interest rate of 8% per annum. Find the simple interest earned.
Solution

Year 0,
Principal = RM1,000

Year 1
Interest for first year = 1,000 \times 0.08 \times 1 = RM80
Simple amount = RM1,080

Year 2
Interest for second year = 1,000 \times 0.08 \times 1 = RM80
Simple amount = RM1,160
Interest earned = RM1,160 – RM1,000 = RM160
OR
I = Prt = 1,000 \times 0.08 \times 2 = RM160
The *simple amount* is the sum of the original *principal* and the *interest* earned.

The simple amount formula is given as

\[ S = P(1 + rt) \]

where \( S = \text{simple amount} \)
Example 2

• RM10,000 is invested for 4 years 9 months in a bank earning a simple interest rate of 10% per annum. Find the simple amount at the end of the investment period.
Solution

- Here, $P = RM10,000$, $r = 10\%$, $t = 4.75\text{ years}$
- From $I = Prt$, we get
  $I = 10,000 \times 0.1 \times 4.75 = RM4,750$
  Simple interest earned is $RM4,750$.
- From $S = P + I$, we get simple amount
  $S = RM10,000 + RM4,750 = RM14,750$
  OR
- From $S = P(1 + rt)$, we get
  $S = 10,000 \times 1 + 0.1 \times 4.75) = RM14,750$. 
Example 3

• Raihan invests RM5,000 in an investment fund for three years. At the end of the investment period, his investment will be worth RM6,125.

Find the simple interest rate that is offered.
Solution

• Here, \( P = \text{RM5,000}, \)
  \( I = \text{RM6,125} - \text{RM5,000} = \text{RM1,125}, \)
  \( t = 3 \text{ years} \)

  From \( I = Prt, \) we get
  \( 1,125 = \text{RM5,000} \times r \times 3 \)
  \( r = 7.5\% \)
Example 4

How long does it take a sum of money to triple itself at a simple interest rate of 5% per annum?
Solution

• Let the original principal be RM\(K\) and time taken be \(t\) years.

Hence interest earned is \(RM3K - RMK = RM2K\).

Then from \(I = Prt\), we get

\[2K = K \times 0.05 \times t\]

\(t = 40\) years
Example 5

Twenty-four months ago, a sum of money was invested. Now the investment is worth RM12,000. If the investment is extended for another twenty-four months, it will become RM14,000. Find the original principal and the simple interest rate that was offered.
Solution

Let original principal = RM P
simple interest rate = r% per annum

<table>
<thead>
<tr>
<th>Year</th>
<th>P</th>
<th>12,000</th>
<th>14,000</th>
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<td>2</td>
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</table>

From the diagram, we derive the following equations.

12,000 = P(1 + 2r)  \hspace{0.5cm} (1)
14,000 = P(1 + 4r)  \hspace{0.5cm} (2)

\frac{(2)}{(1)}: \hspace{0.5cm} \frac{14000}{12000} = \frac{1 + 4r}{1 + 2r}
14,000 (1 + 2r) = 12,000 (1 + 4r)

r = 10%

Substituting r = 10% into equation (1), we get
12,000 = P[1 + 2(10%)]
P = RM10,000
Example 6

Muthu invested RM10,000 in two accounts, some at 10% per annum and the rest at 7% per annum. His total interest for one year was RM820. Find the amount invested at each rate.
Solution

Let amount invested at 10% = RMK
amount invested at 7% = RM(10,000 – K)

Hence,

\[ 0.1K + 700 - 0.07K = 820 \]

\[ K = RM4,000 \]

RM4,000 was invested at 10% and RM6,000 at 7%.
Four basic concepts

1. Exact time: It is the exact number of days between two given dates.
2. Approximate time: It assumes a month has 30 days in the calculation of number of days between two given dates.
3. Ordinary simple interest: In calculating ordinary simple interest, we use a 360-day year.
4. Exact simple interest: This uses a 365/366-day year for interest computation.
Example 7

Find (a) exact time, (b) approximate time, from 15 March to 29 August of the same year.
## Solution

<table>
<thead>
<tr>
<th>Month</th>
<th>Exact time</th>
<th>Approximate time</th>
</tr>
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<tbody>
<tr>
<td>March</td>
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<td>15</td>
</tr>
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<td>April</td>
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<td>July</td>
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<td>30</td>
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<tr>
<td>August</td>
<td>29</td>
<td>29</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>167</strong></td>
<td><strong>164</strong></td>
</tr>
</tbody>
</table>
Example 8

RM1,000 was invested on 15 March 2011. If the simple interest rate offered was 10% per annum, find the interest received on 29 August 2012 using

(a) exact time and exact simple interest,

(b) exact time and ordinary simple interest,

(c) approximate time and exact simple interest,

(d) approximate time and ordinary simple interest.
Solution

(a) Exact time and exact simple interest
\[ I = 1,000 \times 0.1 \times \frac{167}{365} = RM45.75 \]

(b) Exact time and ordinary simple interest (Also called Banker’s rule)
\[ I = 1,000 \times 0.1 \times \frac{167}{360} = RM46.39 \]

(c) Approximate time and exact simple interest
\[ I = 1,000 \times 0.1 \times \frac{164}{365} = RM44.93 \]

(d) Approximate time and ordinary simple interest
\[ I = 1,000 \times 0.1 \times \frac{164}{360} = RM45.56 \]
The present value is the value in today’s money of a sum of money to be received in the future.

From the formula \( S = P(1 + rt) \), we find \( P \) to find the present value, that is

\[
P = \frac{S}{1 + rt}
\]

\[
P = S(1 + rt)^{-1}
\]
Example 10

Find the present value at 8% simple interest of a debt RM3,000, due in ten months.
Solution

From $P = S(1 + rt)^{-1}$, we get

$P = 3,000(1 + 0.08 \times 10/12)^{-1}$

$P = \text{RM2,812.50}$

Hence, the present value of the debt is RM2,812.50.
An equation that states the equivalence of two sets of dated values at a stated date is called an equation of value or equivalence.

The stated date is called the **focal date, the comparison date or the valuation date.**

To set up and solve an equation of value,

1. Draw a time diagram with all the dated values.
2. Select the focal date.
3. Pull all the dated values to the focal date using the stated interest rate.
4. Set up the equation of value and then solve.
Example 11

• A debt of RM800 due in four months and another of RM1,000 due in nine months are to be settled by a single payment at the end of six months. Find the size of this payment using

(a) the present as the focal date,

(b) the date of settlement as the focal date,

assuming money is worth 6% per annum simple interest.
(a) Let the single payment at the end of six months be RM X.

Focal date

\[ \begin{array}{c} 0 \quad 4 \text{m} \quad 6\text{m} \quad 9\text{m} \\ 800 \quad X \quad 1000 \end{array} \]

Amount of the RM800 debt at the focal date
\[ = 800(1 + 0.06 \times 4/12)^{-1} = \text{RM784.31} \]

Amount of the RM1,000 debt at the focal date
\[ = 1,000(1 + 0.06 \times 9/12)^{-1} \]
\[ = \text{RM956.94} \]
Amount of the single payment at the focal date = X(1 + 0.06 \times 6/12)^{-1}

= RM0.97087X

Setting up the equation of value, we get

0.97087X = 784.31 + 956.94

X = RM1,793.49
Amount of the RM800 debt at the focal date

\[= 800(1 + 0.06 \times \frac{2}{12}) = \text{RM808}\]

Amount of the RM1,000 debt at the focal date

\[= 1,000(1 + 0.06 \times \frac{3}{12})^{-1} = \text{RM985.22}\]
Let $X =$ amount of payment at the end of six months. Then,

$$X = RM808 + RM985.22$$

$$= RM1,793.22$$
1. Simple interest I is given by the product of principal, interest rate and time.

\[ I = Prt \]

where \( P \) = principal
\( r \) = simple interest rate
\( t \) = time

If \( r \) is in per annum basis, \( t \) must be in years. If \( r \) is in per month basis, \( t \) must be in months.

\( P, r \) or \( t \) can be determined if \( I \) is given.

\[ P = \frac{I}{rt}, \quad r = \frac{I}{Pt}, \quad t = \frac{I}{Pr} \]
2. If the period is given in terms of two dates, then we have
   (a) exact time = exact number of days between the two given dates
   (b) approximate time = number of days between the two given dates assuming that 1 month = 30 days
   (c) ordinary simple interest which uses 360-day year
   (d) exact simple interest which uses 365/366-day year

3. The simple amount formula is
   \[ S = P(1 + rt). \]

4. Present value, \( P = S/(1+rt) \) or
   \[ P = S(1 + rt)^{-1} \]

5. Equation of value: At the focal date,
   value of debts = value of payments.
2.3 Compound interest

- In this subchapter, we will discuss the time value of money and the uses of compound interest in various financial fields.
- We can apply these concepts to the valuation of different securities, loans, instalment purchases, savings, insurance and investments.
LEARNING OBJECTIVES

By the end of this chapter, you should be able to

• explain the concepts of time value of money;
• derive the compound amount formula;
• use the compound amount formula to find the future value, compound interest, and present value of investments and loans;
• determine the effective interest rate and nominal rate;
• establish the relationship between effective rates and nominal rates;
• establish the relationship between two nominal rates, and
• use the equation of values to solve problems relating to investments and loans.
2.3 Compound interest

• Compound interest computation is based on the principal, which changes from time to time.
• Interest that is earned is compounded or converted into principal and earns interest thereafter.
• Two differences between simple interest and compound interest:
  • Simple interest is based on the original principal while compound interest is based on the principal which grows from one interest interval to another,
  • The simple amount function is a linear function with respect to time while the compound amount function is an exponential function.
Example 1

RM1,000 is invested for three years. Find the interest received at the end of the three years if the investment earns 8% compounded annually.
Solution

• Year 0  Principal = RM1,000
  Interest for first year = 1,000 × 0.08 × 1 = RM80
• Year 1:  Amount at the end of first year = RM1,000 + RM80 = RM1,080
  Interest for second year = 1,080 × 0.08 × 1 = RM86.40
• Year 2:  Amount at the end of second year = RM1,080 + RM86.40 = RM1,166.40
  Interest for third year = 1,166.40 × 0.08 × 1 = RM93.31
• Year 3:  Amount at the end of third year = RM1,166.40 + RM93.31 = RM1,259.71
• Compound interest earned = amount – original principal = RM1,259.71 – RM1,000
  = RM259.71
• Note that interest computed for each year is based on
  the principal which changes every year.
Some important terms

- **Original principal**: The original principal, denoted by P, is the original amount invested.

- **Annual nominal rate**: Annual nominal rate, denoted by k, is the interest rate for a year together with the frequency in which interest is calculated in a year. If interest is compounded twice a year, it is said to be compounded semi-annually.

- **Interest period**: Interest period is the length of time in which interest is calculated.

- **Frequency of conversions**: Frequency of conversions, denoted by m, is the number of times interest is calculated in a year. In other words, it is the number of interest periods in a year.

- **Periodic interest rate**: Periodic interest rate, denoted by i, is the interest rate for each interest period.

- **Number of interest periods in the investment period**: The number of times interest is calculated in the total investment period.
Compound interest formula

- Future value $S$ of an investment $P$ after $n$ interest periods is $S = P(1 + i)^n$
- The factor $(1 + i)^n$ is called the future value of 1 at the rate $i$ per interest period for $n$ interest periods.
- The compound interest $I$ is the difference between the future value and the original principal, that is $I = S - P$
Consider Example 1

RM1,000 is invested for three years. Find the interest received at the end of the three years if the investment earns 8% compounded annually.
Solution

• Applying the future value formula, we have
  \[ S = P (1 + i)^n \]
  where \( n = 3 \) for three years as interest rate is 8% compounded annually.
  Thus, \[ S = 1,000 \times (1 + 0.08)^3 = RM1,259.71 \]

• Compound interest \( I \) is
  \[ I = RM1,259.71 - RM1,000 = RM259.71 \]
Example 2

Find the future value of RM1,000 which was invested for
(a) 4 years at 4% compounded annually,
(b) 5 years 6 months at 14% compounded semi-annually,
(c) 2 years 3 months at 4% compounded quarterly,
(d) 5 years 7 months at 5% compounded monthly,
(e) 2 years 8 months at 9% compounded every 2 months,
(f) 250 days at 10% compounded daily.
Solution

(a) \( S = 1,000 \times (1 + 4\%)^4 = RM1,169.86 \)
(b) \( S = 1,000 \times (1 + 14\%/2)^{11} = RM2,104.85 \)
(c) \( S = 1,000 \times (1 + 4%/4)^9 = RM1,093.69 \)
(d) \( S = 1,000 \times (1 + 5%/12)^{67} = RM1,321.26 \)
(e) \( S = 1,000 \times (1 + 9%/6)^{16} = RM1,268.99 \)
(f) \( S = 1,000 \times (1 + 10%/360)^{250} = RM1,071.90 \)
Example 3

RM9,000 is invested for 7 years 3 months. This investment is offered 12% compounded monthly for the first 4 years and 12% compounded quarterly for the rest of the period. Calculate the future value of this investment.
12% compounded monthly

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
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<th>6</th>
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</thead>
<tbody>
<tr>
<td>9000</td>
<td>n=48</td>
<td>$S_4$</td>
<td>n=13</td>
<td>$S_{7.25}$</td>
<td></td>
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</tbody>
</table>

Amount of investment at the end of 4 years,

$$S_4 = P(1 + i)^n = 9,000 \ (1 + 12%/12)^{48}$$

$$= RM14,510.03$$

Amount of investment at the end of 7.25 years,

$$S_7 = P(1 + i)^n = 14,510.03 \ (1 + 12%/4)^{13}$$

$$= RM21,308.48$$
Example 4

Lolita saved RM5,000 in a savings account which pays 12% interest compounded monthly. Eight months later she saved another RM5,000. Find the amount in the account two years after her first saving.
Amount = 5,000(1 + 1\%)^{24} + 5,000(1 + 1\%)^{16}

= RM12,211.57
EXAMPLE 5

What is the nominal rate compounded monthly that will make RM1,000 become RM2,000 in five years?
Solution

From $S = P(1 + i)n$, we get

$2,000 = 1,000 \cdot (1 + \frac{k}{12})^{60}$

$2 = (1 + \frac{k}{12})^{60}$

$2^{1/60} = 1 + \frac{k}{12}$

$k = 13.94\%$
Example 6

How long does it take a sum of money to double itself at 14% compounded annually?
Solution

Let the original principal = \( W \)
Therefore sum after \( n \) years = \( 2W \)
From \( S = P(1 + i)^n \), we get
\[
2W = W(1 + 14\%)^n
\]
\[
2 = (1 + 14\%)^n
\]
\[
\text{lg } 2 = n \text{ lg } 1.14
\]
\[
n = 5.29 \text{ years}
\]
2.3.1 Effective, nominal and equivalent rates

• Two rates are equivalent if they yield the same future value at the end of one year.
• A nominal rate is one in which interest is calculated more than once a year.
• An annual effective rate (or effective rate) is the actual rate that is earned in a year.
• Effective rate can also be defined as the simple interest rate earned in a year.
Example 7

RM100 is invested for one year. If the interest rate is (a) 9.04% compounded annually, (b) 8.75% compounded quarterly, find the amount after one year.
Solution

(a) From $S = P(1 + i)^n$, we get
Amount = $100(1 + 9.04\%)^1 = RM109.04$

(b) From $S = P(1 + i)^n$, we get
Amount = $100 (1+ 8.75\%)^4 = RM109.04$

It should be noted that

• 9.04% compounded annually is an effective rate.
• 8.75% compounded quarterly is a nominal rate.
• 9.04% compounded annually is equivalent to 8.75% compounded
The relationship between the nominal rate and effective rate is derived as follows. Assume a sum RM \( P \) is invested for one year. Then the future value after one year

(a) at \( r \)% effective = \( P(1 + r) \)

(b) at \( k \)% compounded \( m \) times a year = \( P(1 + k/m)^m \)

Equating the future values in (a) and (b), we obtain

\[ P(1 + r) = P(1 + k/m)^m \]

\[ 1 + r = (1 + k/m)^m \]
Example 8

Find the effective rate which is equivalent to 16% compounded semiannually.

Solution

\[ 1 + r = (1 + \frac{k}{m})^m \]

\[ r = (1 + \frac{16\%}{2})^2 - 1 \]

\[ = 16.64\% \]
Example 9

Find the nominal rate, compounded monthly which is equivalent to 9% effective rate.

Solution

\[ 1 + r = (1 + k/m)^m \]

\[ 1.09 = (1 + k/12)^{12} \]

\[ (1.09)^{1/12} = 1 + k/12 \]

\[ 1.007207 = 1 + k/12 \]

\[ k = 8.65\% \]
Example 10

Kang wishes to borrow some money to finance some business expansion. He has received two different quotes:

Bank A: charges 15.2% compounded annually.
Bank B: charges 14.5% compounded monthly.

Which bank provides a better deal?
Solution

Effective rate of bank B = \((1 + 14.5\%/12)^{12} - 1\) 
= 15.5%

Bank A charges 15.2% effective rate.

• Hence, bank A provides a better deal as it charges a lower effective rate.
Example 11

Desmond borrowed a certain sum of money at 10% per annum simple interest. The loan was repaid one year later with the interest charged.

What was the effective rate of the loan?

Solution

Note that the effective rate can be defined as the simple interest rate earned in a year. Hence the effective rate was 10%.
Relationship between two nominal rates

- The relationship between two nominal rates is given as follows.

\[(1 + \frac{k}{m})^m = (1 + \frac{K}{M})^M\]

where \(k\) and \(K\) are two different annual rates with two different frequencies of conversions, \(m\) and \(M\), respectively.
Example 12

Find \( k \% \) compounded quarterly which is equivalent to \( 6 \% \) compounded monthly.

Solution

\[
(1 + \frac{k}{m})^m = (1 + \frac{K}{M})^M
\]

\[
(1 + \frac{k}{4})^4 = (1 + \frac{6\%}{12})^{12}
\]

\[
(1 + \frac{k}{4}) = (1 + \frac{6\%}{12})^3
\]

\[
k = 6.03\%
\]
2.3.2 Present value

- The **present value (or discounted value)** at \(i\)% per interest period of an amount \(S\) due in \(n\) interest periods is that value \(P\) which will yield the sum \(S\) at the same interest rate after \(n\) interest periods.

- Hence from \(S = P (1 + i)^n\), we get
  - \(P = S/(1 + i)^n\)
  - \(P = S \cdot (1 + i)^{-n}\)

- The process of finding the present or discounted value is called **discounting**.
Example 13

A debt of RM3,000 will mature in three years’ time. Find

(a) the present value of this debt,
(b) the value of this debt at the end of the first year,
(c) the value of this debt at the end of four years, assuming money is worth 14% compounded semi-annually.
Solution

Year 0 1 2 3 4

| P₀ | P₁ | 3000 |

(a) From \( P = S(1 + \frac{r}{2})^n \), we get

\[ P₀ = 3000 \left(1 + \frac{14\%}{2}\right)^2 = RM1,999.03 \]

(b) \( P = S(1 + \frac{r}{2})^n \)

\[ P₁ = 3000 \left(1 + \frac{14\%}{2}\right)^4 = RM2,288.69 \]

(c) Here, we have to find S instead of P in the formula, \( S = P \left(1 + \frac{r}{2}\right)^n \)

as the value of the debt to be determined is on the right side of the
original debt.

From \( S = P(1 + \frac{r}{2})^n \), we get

\[ S = 3000 \left(1 + \frac{14\%}{2}\right)^2 = RM3,434.70 \]
Example 14

A project which requires an initial outlay of RM9,000 will produce the following annual inflows.

Year 1 RM2,000

Year 2 RM4,000

Year 3 RM6,000

What is the net present value, NPV if the discount rate is 8% per annum?
Solution

• Present value of inflows
  \[\frac{2000}{1 + 8\%} + \frac{4000}{(1 + 8\%)^2} + \frac{6000}{(1 + 8\%)^3}\]

• Present value of outflows = 9,000

NPV = Present value of inflows – Present value of outflows
  \[= \frac{2000}{1 + 8\%} + \frac{4000}{(1 + 8\%)^2} + \frac{6000}{(1 + 8\%)^3} - 9000\]
  \[= RM1,044.20\]

• A positive net present value indicates that the project is viable whereas a negative net present value indicates that the project will incur loss if it is initiated.
An equation of value is an equation that expresses the equivalence of two sets of obligations at a focal date.

What is owed = What is owned at the focal date
or

What is given = What is received at the focal date

In computing equation of value using the compound interest rate, the two sets of obligations are the same no matter where we put the focal point.
Example 15

- A debt of RM7,000 matures at the end of the second year and another of RM8,000 at the end of six years. If the debtor wishes to pay his debts by making one payment at the end of the fifth year, find the amount he must pay if money is worth 6% compounded semi-annually using

(a) the present as the focal date,
(b) the end of the fifth year as the focal date.
Solution

- (a) Let the payment be RM X.

<table>
<thead>
<tr>
<th>Focal date</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tbody>
</table>

Formulating the equation of value at the focal date as shown, we get
What is owed = What is owned

\[ X \left(1 + 3\%\right)^{-10} = 7,000\left(1 + 3\%\right)^{-4} + 8,000\left(1 + 3\%\right)^{-12} \]

\[ X = \text{RM15,899.13} \]
(b) Let the payment be RM X.

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7000 &  & X & & & & 8000 \\
\end{array}
\]

Formulating the equation of value at the focal date as shown, we get

What is owed = What is owned

\[
X = 7,000(1 + 3\%)^6 + 8,000(1 + 3\%)^{-2}
\]

\[
= \text{RM}15,899.13
\]
Example 16

A debt of RM7,000 matures at the end of the second year and another of RM8,000 at the end of six years. If the debtor wishes to pay his debts by making two equal payments at the end of the fourth year and the seventh year, what are these payments assuming money is worth 6% compounded semi-annually?
Solution

- Let the payment be RM \( X \) each.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \text{ years} \\
7000 & X & 8000 & X
\end{array}
\]

- Formulating the equation of value at the focal date as shown, we get

\[
\text{What is owed} = \text{What is owned}
\]

\[
X(1 + 3\%)^6 + X = 7000(1 + 3\%)^{10} + 8000(1 + 3\%)^2
\]

\[
X = \text{RM}8,155.97
\]
Example 17

Roland invested RM10,000 at 12% compounded monthly. This investment will be given to his three children when they reach 20 years old. Now his three children are 15, 16 and 19 years old. If his three children will receive equal amounts, find the amount each will receive.
Solution

- Let the payments be RM X each.

  Focal date
  ——— ——— ——— ——— ——— ———
  0 1 4 5

  X X X

- Formulating the equation of value at the focal date as shown, we get

  What is given = What is received

  \[ 10,000 = X(1 + 1\%)^{-12} + X(1 + 1\%)^{-48} + X(1 + 1\%)^{-60} \]

  \[ X = RM4,858.71 \]
2.3.3 Continuous compounding

- Future value of a sum of money $P$ compounded continuously is given by

$$S = Pe^{it}$$

where $S$ = future value

$P$ = original principal

e = 2.718282...

$i$ = continuous compounding rate

t = time in years
Example 18

Find the accumulated value of RM1,000 for six months at 10% compounded continuously.

• **Solution**

Here, \( P = RM1,000 \)

\[ i = 10\% \]

\[ t = 0.5 \text{ year} \]

From \( S = Pe^{it} \), we get

\[ S = 1,000 \left[ e^{10\% \times 0.5} \right] \]

\[ = RM1,051.27 \]
Example 19

Find the amount to be deposited now so as to accumulate RM1,000 in 18 months at 10% compounded continuously.

• Solution

Here, \( S = RM1,000 \)

\( i = 10\% \)

\( t = 1.5 \text{ years} \)

From \( S = Pe^{it} \), we get

\[ 1,000 = Pe^{10\% \times 1.5} \]

\( P = RM860.71 \)
SUMMARY

1. The future value, S of principal, P after n interest periods at compound interest rate, i per interest period is \( S = P(1 + i)^n \)
   where \( S \) = future value or amount
   i = interest rate per interest period
   n = number of interest periods

2. **Compound interest, I is given by**
   \( I = S - P \)

3. If \( r \) is the **effective interest rate** that is equivalent to nominal rate \( k\% \) compounded \( m \) times a year, then
   \( 1 + r = (1 + k/m)^m \)

4. The relationship between two nominal rates of interest is given by
   \( (1 + k/m)^m = (1 + K/M)^M \)
   where \( k \) and \( K \) are the two different nominal rates with their respective frequencies of conversion, \( m \) and \( M \) per year.
5. The **present value** at $i\%$ per interest period of an amount $S$ due in $n$ interest periods is

\[ P = \frac{S}{(1 + i)^n} \]

or

\[ P = S (1 + i)^{-n} \]

6. **Equation of value**

At the focal date,

value of debts = value of payments, or

what is owed = what is owned
2.4 Annuity

- Annuity is a series of (usually) equal payments made at (usually) equal intervals of time.
- Examples of annuity are shop rentals, insurance policy premiums, annual dividends received and instalment payments.
- In each case, equal payments are deposited or paid periodically over time.
- **In ordinary annuity certain** payments are made at the end of the payment period and the interest and payment period are of the same interval.
LEARNING OBJECTIVES

• By the end of this chapter, you should be able to
  ✔ explain the term ordinary annuity certain;
  ✔ derive the future value and the present value of ordinary annuity certain;
  ✔ find the future value of annuity;
  ✔ find the present value of annuity;
  ✔ solve for annuity payment, R, the number of payments, n, and the interest rate, i, and
  ✔ identify the problems where the present value and the future value of annuity formulae can be appropriately applied.
2.4.1 Future value of ordinary annuity certain

- **Future value (or accumulated value)** of an ordinary annuity certain is the sum of all the future values of the periodic payments.

Let periodic payments = $R$

interest rate per interest period = $i\%$

term of investment = $n$ interest periods

future value of annuity at end of $n$ interest periods = $S$
$S = R(1 + i)^{n-1} + R(1 + i)^{n-2} + R(1 + i)^{n-3} + \ldots + R(1 + i)^2 + R(1 + i) + R$

$S = R \left[ 1 + (1 + i)^1 + (1 + i)^2 + (1 + i)^3 + \ldots + (1 + i)^{n-3} + (1 + i)^{n-2} + (1 + i)^{n-1} \right]$

The right-hand side expression inside the brackets is a geometric series with 1 as the first term and $(1 + i)$ as the common ratio. Summing up all the terms, we obtain

$(1 + i)^n - 1$

$S = R \left[ \frac{(1 + i)^n - 1}{i} \right] \quad \text{or} \quad i$

$S = Rs_{n,i} \quad \text{with} \quad s_{n,i} = \frac{(1 + i)^n - 1}{i}$

The expression, $s_{n,i} = \frac{(1 + i)^n - 1}{i}$, is the future value of annuity of RM1 per payment interval for $n$ intervals. $s_{n,i}$ is read as ‘$s$ angle $n$ at $i$’ and its value can be found for certain $i$ and $n$ in the annuity amount table (See Appendix).

- Interest earned, $I$, from investing in annuity is given by $I = S - nR$. 
Example 1

RM100 is deposited every month for 2 years 7 months at 12% compounded monthly. What is the future value of this annuity at the end of the investment period? How much interest is earned?
Solution

Here, \( R = \text{RM100} \), \( i = 12\%/12 = 1\% \), \( n = 2 \times 12 + 7 = 31 \)

From \( S = R[(1 + i)^n - 1]/i \), we get

\[
S = 100 \cdot \frac{(1 + 0.01)^{31} - 1}{0.01} = \text{RM3,613.27}
\]

Alternatively by using tables, we get

\[
S = \text{Rs}_{n1} = 100 \cdot s_{31.1\%} = 100 \times 36.13274045 = \text{RM3,613.27}
\]

Interest earned, \( I = S - nR = 3,613.27 - (31 \times 100) = \text{RM513.27} \)
Example 2

RM100 is deposited every 3 months for 2 years and 9 months at 8% compounded quarterly. What is the future value of this annuity at the end of the investment period? How much interest is earned?
Solution

- Here, $R = \text{RM}100$, $i = 8\%/4 = 2\%$
  $n = 2 \times 4 + 3 = 11$
  From $S = R[(1 + i)^n - 1]/i$, we obtain
  $S = 100[(1 + 0.02)^{11} - 1]/0.02$
  $= \text{RM}1,216.87$

- Alternatively by using tables, we get
  $S = Rs_{n\,i} = Rs_{11\,2\%}$
  $= 100 \times 12.16871542 = \text{RM}1,216.87$
  Interest earned, $I = S - nR = 1,216.87 - (11 \times 100)$
  $= \text{RM}116.87$
Example 3

RM100 was invested every month in an account that pays 12% compounded monthly for two years. After the two years, no more deposit was made. Find the amount of the account at the end of the five years and the interest earned.
This is an example of a **forborne annuity** in which the annuity earns interest for two or more periods after the last payment is made.

Amount in the account just after 2 years

\[ S = R \left[ \frac{(1 + i)^n - 1}{i} \right] \]

\[ = 100 \left[ (1 + 0.01)^{24} - 1 \right] / 0.01 \]

\[ = \text{RM}2,697.35 \]

Amount in the account at the end of 5 years

\[ S = P(1 + i)^n \]

\[ = 2,697.35 \times (1 + 1\%)^{36} \]

\[ = \text{RM}3,859.28 \]

Interest earned, \[ I = S - nR = 3,859.28 - (24 \times 100) \]

\[ = \text{RM}1,459.28 \]
Example 4

Lily invested RM100 every month for five years in an investment scheme. She was offered 5% compounded monthly for the first three years and 9% compounded monthly for rest of the period. Find the accumulated amount at the end of the five years. Hence, determine the interest earned.
Solution

• Referring to the diagram above, we see that the annuity of RM100 was offered at two different rates of interest. Since there are two different rates, we break up the annuity into two streams of annuity as shown below.

• *Stream 1*

  Amount just after the 3rd year = 100\[(1+0.05/112)^{36} - 1\]/(0.05/12)

  = RM3,875.33

  Amount at the end of 5 years = 3,875.33(1 + 0.09/12)/(0.09/12)

  = RM4,636.50
• Stream 2

Amount at the end of 5 years

\[= 100[(1 + 0.09)^{24} - 1]/(0.09/12)\]

\[= RM2,618.85\]

• Hence, the amount in the account at the end of 5 years is RM4,636.50 + RM2,618.85 = RM7,255.35.

• Interest earned, \(I = S - nR\)

\[= 7,255.35 - (60 \times 100)\]

\[= RM1,255.35\]
Example 5

Table shows the monthly deposits that were made into an investment account that pays 12% compounded monthly.

<table>
<thead>
<tr>
<th>Year</th>
<th>Monthly deposits</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007</td>
<td>RM100</td>
</tr>
<tr>
<td>2008</td>
<td>RM200</td>
</tr>
<tr>
<td>2009</td>
<td>RM300</td>
</tr>
</tbody>
</table>

Find the value of this investment at the end of 2009. Find also the interest earned.
Solution

Stream 1

Value of the annuity of RM100 at end of 2012

\[ \text{Value} = \frac{100 \times (1 + 0.01)^{12} - 1}{0.01} \times (1 + 0.01)^{24} = RM1,610.34 \]

Stream 2

Value of the annuity of RM200 at end of 2012

\[ \text{Value} = \frac{200 \times (1 + 0.01)^{12} - 1}{0.01} \times (1 + 0.01)^{12} = RM2,858.19 \]

Stream 3
• Stream 3
  Value of the annuity of RM300 at end of 2012
  \[= 300[(1 + 0.01)^{12} - 1]/0.01 = RM3,804.75\]
• Hence, the value of the investment account at the end of 2012 is
  \[RM1,610.34 + RM2,858.19 + RM3,804.75 = RM8,273.28\]
• Interest earned, \(I = S - nR\)
  \[= 8,273.28 - [12 \times 100 + 12 \times 200 + 12 \times 300]\]
  \[= RM1,073.28\]
Example 6

RM300 was invested every month in an account that pays 10% compounded annually for 18 months. Calculate the amount in the account after 18 months. How much interest was earned?
Solution

• This is an example of general annuity where the payment period (one month) differs from the interest period (one year). To solve this problem, one way is to change the effective rate, $r$ to a nominal rate compounded monthly, $k$.

$$1 + r = (1 + k/m)^m$$

$$1 + 10\% = (1 + k/12)^{12}$$

Let $i = k/12$, where $i$ is the interest rate per interest period.
Solution continue

Then, $1 + 10\% = (1 + i)^{12}$

- Solving for \( i \), we get
  \[ i = (1.1)^{1/12} - 1 = 0.0079741 \]

Amount in the account = \[ 300[(1 + i)^{18} - 1] / i \]
= \[ 300[(1 + 0.0079741)^{18} - 1] / 0.0079741 \]
= RM5,782.05

- Interest earned, \( I = S - nR \)
  = RM5,782.05 - (18 \times 300)
  = RM382.05
2.4.2 Present value of an ordinary annuity certain

The present value (discounted value) of an ordinary annuity certain is the sum of all present values of the periodic payments. Let

periodic payments = R, interest rate per interest period = i\%, term of investment = n interest periods,

present value of annuity at end of n interest periods = A

Present value of annuity,
\[ A = R(1 + i)^{-1} + R(1 + i)^{-2} + R(1 + i)^{-3} + \ldots + R(1 + i)^{-(n-1)} + R(1 + i)^{-n} \]
\[ A = R[(1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3} + \ldots + (1 + i)^{-(n-1)} + (1 + i)^{-n}] \]

Note that the right-hand side expression inside the brackets is a geometric series with \((1 + i)^{-1}\) as the first term and \((1 + i)^{-1}\) as the common ratio. Summing up all the terms, we obtain
\[ A = R[1 - r^n]/r \]
\[ A = R[1 - (1 + i)^{-n}]/[1-(1+i)] \]
\[ A = R[1 - (1 + i)^{-n}]/i = R a_{n,i} \]
\[ a_{n,i} = [1 - (1 + i)^{-n}]/i \]

Interest paid, I = nR – A.

The expression, \( a_{n,i} = [1 - (1 + i)^{-n}]/i \) is the present value of an annuity of RM1 per payment interval for n intervals. The symbol \( a_{n,i} \) is read as ‘a angle n at i’ and its value can be found for certain i and n in the present value annuity table.
Example 7

- Raymond has to pay RM300 every month for 24 months to settle a loan at 12% compounded monthly.
(a) What is the original value of the loan?
(b) What is the total interest that he has to pay?
Solution

- (a)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>....</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>300</td>
<td>300</td>
<td>300</td>
<td>300</td>
<td>...</td>
<td>300</td>
<td>300</td>
<td>300</td>
</tr>
</tbody>
</table>

A

No payment                              Last payment

Here, R = RM300, i = 12%/12 = 1%, n = 24

From \( A = R \left[1 - (1 + i)^{-n}\right] / i \), we get

\( A = 300 \left[1 - (1 + 1%)^{-24}\right] / 1\% = RM6,373.02 \)

- Alternatively by using tables, we get

\( A = Ra_{n1} = 300a_{24}^{1\%} = 300(21.24338726) = RM6,373.02 \)

- (b) Total interest = (300 \times 24) - RM6,373.02

\[ = RM826.98 \]
Example 8

John won an annuity that pays RM1,000 every three months for three years. What is the present value of this annuity if money is worth 16% compounded quarterly?
Solution

• Here, \( R = RM1,000 \), \( i = 16\%/4 = 4\% \)
\( n = 4 \times 3 = 12 \)
From \( A = R[1 - (1 + i)^{-n}] / i \), we get
\( A = 1,000[1 - (1 + 4\%)^{-12}] / 4\% \)
= RM9,385.07

• Or we can use present value table.
\( A = 1000(9.38507376) = RM9,385.07 \)
Example 9

James intends to give a scholarship worth RM5,000 every year for six years. How much must he deposit now into an account that pays 7% per annum to provide the scholarship?
Solution

- Here, $R = RM5,000$, $i = 7\%$, $n = 6$
  
  From $A = R \frac{1 - (1 + i)^{-n}}{i}$, we get
  
  $A = 5,000 \frac{1 - (1 + 7\%)^{-6}}{7\%}$
  
  $= RM23,832.70$

- Alternatively by using tables, we get
  
  $A = Ra_{n \cdot i} = 5,000a_{6 \cdot 7\%}$
  
  $= 5,000 (4.76653966)$
  
  $= RM23,832.70$
Example 10

Shirley wants to provide a scholarship of RM3,000 each year for the next three years. The scholarship will be awarded at the end of each year to the best student. If the money is worth 10% compounded annually, find the amount that must be invested now.
Solution

• From $A = R[1 - (1 + i)^{-n}]/i$, we get
  
  $A = 3,000[1 - (1 + 10\%)^{-3}]/10\%$
  
  $= \text{RM}7,460.56$

  Shirley must invest RM7,460.56 now
Example 11

Under a contract, Jenny has to pay RM100 at the beginning of each month for 15 months. What is the present value of the contract if money is worth 12% compounded monthly? Find the interest paid by Jenny.
This is an example of **annuity due** where payment is made at the beginning of the payment period.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

\[ i = \frac{12\%}{12} = 1\% = 0.01 \]

No payment

Present value of the contract

\[ = 100 + 100[1-(1+0.01)^{-14}]/0.01 \]

\[ = RM1,400.37 \]

Interest paid, \( I = nR - A = (15 \times 100) - 1,400.37 \)

\[ = RM99.63 \]
Example 12

Find the present value of an annuity of RM500 every year for 5 years if the first payment is made in 2 years. Assume that money is worth 6% compounded annually.
Solution

- This is an example of a deferred annuity in which the first payment is made at some later date.

Value of the annuity at year 1

\[ A_1 = 500 [1 - (1 + 6\%)^{-5}] / 6\% = RM2,106.18 \]

Value of \( A_1 \) at year 0 = \( A(1 + i)^{-n} \)

\[ = 2,106.18 \times (1 + 6\%)^{-1} \]
\[ = RM1,986.96 \]

Hence, the present value of the annuity is RM1,986.96.
Example 13

Good Fortune Industries decided to pay back a RM20,000 loan by making six semi-annual payments at 12% compounded semi-annually.

(a) Find the semi-annual payment.

(b) Find the interest and principal portions of the first payment.

(c) What is the outstanding balance after the first payment?

(d) Find the interest and principal portions of the second payment.

(e) What is the outstanding balance after the second payment?
Solution

(a) Here, \( A = RM20,000, \ i = 6\%, \ n = 6 \)

From \( A = R \left[ 1 - (1 + i)^{-n} \right]/i \), we get

\[
20,000 = R \left[ 1 - (1 + 6\%)^{-6} \right] / 6\%
\]

\[
R = RM4,067.25
\]

The semi-annual payment is RM4,067.25.

(b) Interest due after first payment = 20,000 \times 12\% \times \frac{1}{2} = RM1,200

Principal payment = RM4,067.25 – RM1,200 = RM2,867.25

The interest and principal portions are RM1,200 and RM2,867.25 respectively.

(c) Outstanding balance after the first payment = RM20,000 – RM2,867.25

= RM17,132.75

(d) Interest due after second payment = 17,132.75 \times 12\% \times \frac{1}{2} = RM1,027.97

Principal payment = RM4,067.25 – RM1,027.97 = RM3,039.28

The interest and principal portions are RM1,027.97 and RM3,039.28 respectively.

(e) Outstanding balance after the second payment = RM17,132.75 – RM3,039.28

= RM14,093.47
Solving for $R$, $n$ and $i$

• In the following section, we shall discuss how to find the periodic payment, $R$, the number of payments $n$, and the interest rate per interest period, $i$. 
Example 14

Find the amount to be invested every three months at 10% compounded quarterly to accumulate RM10,000 in three years. Find the interest earned.

**Solution**

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>11</th>
<th>12 deposit</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>R</td>
<td>R</td>
<td>…</td>
<td>R</td>
<td>R</td>
<td>R R S = 10,000</td>
</tr>
</tbody>
</table>

Here, \( S = RM10,000 \), \( i = 10\%/4 = 2.5\% \), \( n = 4 \times 3 = 12 \)

From \( S = R(1 + i)^n - 1 \)/\( i \), we get

\[
10,000 = R\left[(1 + 2.5\%)^{12} - 1\right]/2.5\%
\]

\( R = RM724.90 \)

- Alternatively by using tables, we get

\( S = Rs_{a, \text{i}, n} = 12, i = 2.5\% \)

\[
10,000 = R(13.795)
\]

\( R = RM724.90 \)

- Interest earned, \( I = S - nR = 10,000 - (12 \times 724.90) \)

\[
= RM1,301.20
\]
Example 15

Mariam invests RM12,000 in an account that pays 6% compounded monthly. She intends to withdraw an equal amount every month for two years and when she makes her last withdrawal, her account will have zero balance. Find the size of these withdrawals.
Solution

• Here, \( A = \text{RM12,000}, \quad i = 6\%/12 = 0.5\%, n = 2 \times 12 = 24 \)
  From \( A = R\left[1 - (1 + i)^{-n}\right]/i \), we get
  \( 12,000 = R\left[1 - (1 + 0.5\%)^{-24}\right] 0.5\% \)
  \( R = \text{RM531.85} \)

• Alternatively by using tables, we get
  \( A = Ra_{n\ i} \)
  \( 12,000 = Ra_{24\ 0.5\%} \)
  \( 12,000 = R(22.56286622) \)
  \( R = \text{RM531.85} \)
Example 16

Rosalind borrowed RM80,000 at 12% compounded monthly for three years.

(a) Find her monthly payment.

(b) If she had not paid her first 5 monthly payments, how much should she pay on her sixth payment to settle all outstanding arrears?

(c) If she wanted to settle all of the loan immediately after paying the first 5 monthly payments, how much additional payment did she have to make?

(d) If she had made the first 5 monthly payments and wanted to settle all of the loan in the sixth payment, how much should she pay? How much interest was paid?
Solution

• (a) From $S = R\left[(1 + i)^n - 1 \right]/i$, we get
  
  $80,000 = R\left[1 - (1 + 1\%)^{-36}/1\% \right]$
  
  $R = RM2,657.14$ = monthly payment

• (b) Outstanding arrears $S = R\left[(1 + i)^n - 1 \right]/i$
  
  $= 2,657.14\left[(1 + 1\%)^{-6} - 1\right]/1\%$
  
  $= RM16,346.77$

• (c) Outstanding loan $= 2,657.14a_{31\%}$
  
  $= 2,657.14\left[1 - (1 + 1\%)^{-31}/1\% \right]$
  
  $= RM70,526.57$ = additional payment.

• (d) To find the sixth payment.
  
  The sixth payment $= 2,657.14 + 2,657.14a_{30\%}$
  
  $= RM71,231.83$
Example 17

• A RM10,000 used car is bought for RM2,000 down payment, 14 payments of RM500 a month and a final 15th payment. If interest charged is 9% compounded monthly, find the size of the final payment.

• Solution

Let the final payment be K ringgit.

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & \ldots & 13 & 14 & 15 \\
2000 & 500 & 500 & 500 & 500 & \ldots & 500 & 500 & K \\
\end{array}
\]

\[A=10,000\]

\[
10,000 = 2,000 + 500[1- (1 + 9%/12)^{-15}]/(9%/12) + K(1+9%/12)^{-15}
\]

\[K= RM1,541.98\]
Example 18

Joanne purchased a shop and mortgaged it for RM100,000. The mortgage requires repayment in equal monthly payments over ten years at 16% compounded monthly. Just immediately after making the 80th payment, she had the loan refinanced at 14% compounded monthly. What is the new monthly payment if the number of payments remained the same?
Solution

- Let the original monthly payment be $K$ ringgit.
  From $A = R[1 - (1 + i)^{-n}] / i$, we get
  
  $100,000 = K[1 - (1 + 16%/12)^{-120}] / (6%/12)$
  
  $K = RM1,675.13 =$ original monthly payment
  
  Balance outstanding just after the 80th payment
  
  $= 1,675.13[1 - (1 + 6%/12)^{-40}] / 6%$
  
  $= RM51,671.39$
  
- Now let the new monthly payment be $R$. Therefore
  $51,671.39 = R[1 - (1 + 14%)^{-40}] / (14%/12)$
  
  $R = RM1,623.95$
To find the number of payments, \( n \)

**Example 19**

Jimmy has to pay RM443.21 every month to settle a loan at 6% compounded monthly. Find the number of payments he has to make.

**Solution**

- From \( A = R[1 - (1 + i)^{-n}]/i \), we get
  
  \[
  10,000 = 443.21[1 - (1 + 6%/12)^{-n}]/(6%/12)
  \]
  
  \[
  0.112813 = 1 - (1 + 6%/12)^{-n}
  \]
  
  \[
  (1 + 6%/12)^{-n} = 0.887186
  \]
  
  \[
  -n \log (1 + 6%/12) = \log 0.887186
  \]
  
  \[
  n = 24 \text{ payments}
  \]

- Jimmy has to make 24 payments to settle the loan.
Example 20

Roger borrowed RM100,000 at 12% compounded monthly. How many monthly payments of RM2,000 should Roger make? What would be the concluding size of the final payment?
Solution

From $A = R\left[1 - (1 + i)^{-n}\right]/i$, we get

$100,000 = 2,000\left[1 - (1 + 1%)^{-n}\right]/1%$

$0.5 = 1 - (1 + 1%)^{-n}$

$(1.01)^{-n} = 0.5$

$-n \log (1.01) = \log 0.5$

$n = -\log 0.5/\log 1.01$

$n = 69.7$ payments

Roger can either pay 68 payments of RM2,000 each and one concluding payment of more than RM2,000, or 69 full payments and one concluding payment of less than RM2,000.

**Method 1**

68 payments of RM2,000 each and a final payment of more than RM2,000 on the 69th payment.

Let the concluding payment be $K$ ringgit.

$100,000 = 2,000 \left[1 - (1 + 1%)^{-68}\right]/1% + K(1 + 1%)^{-69}$

$K = RM3,310.56$

**Method 2**

69 payments of RM2,000 each and a final payment of less than RM2,000 on the 70th payment.

$100,000 = 2,000\left[1 - (1 + 1%)^{-69}\right]/1% + K(1 + 1%)^{-70}$

$K = RM1,323.66$
Example 21

Betty bought a TV with a cash prize of RM6,500 by making an initial deposit of RM2,000. The balance will be settled by making 18 monthly deposits of RM300 each. Find

(a) the nominal rate compounded monthly that is being charged,

(b) the effective rate that is being charged.
Solution

Cash balance = cash price – deposit

= RM6,500 – RM2,000 = RM4,500

From \( A = R[1 - (1 + i)^{-n}] / i \), we get

\[ 4,500 = 300[1 - (1 + i)^{-18}] / i \]

\[ 4,500 = 300a_{18}i \]

\[ a_{18}i = 15 \]

From tables, we get

\[ a_{18 \, 1.75\%} = 15.3269 \]

\[ a_{18 \, 2\%} = 14.9920 \]

Using linear interpolation, we get

\[ (i - 1.75\%)/(2\% - 1.75\%) = (15 - 15.3269)/(14.9920 - 15.3269) \]

\[ i = 1.994\% \]
(a) Nominal rate compounded monthly

\[ = 12i = 12(0.01994) \]

\[ = 23.93\% \]

(b) Effective rate \( = (1 + i)^{12} - 1 \)

\[ = (1 + 23.93\%/12)^{12} - 1 \]

\[ = 26.74\% \]
2.4.3 Amortization

An interest bearing debt is said to be amortized when all the principal and interest are discharged by a sequence of equal payments at equal intervals of time.
Amortization schedule

An amortization schedule is a table showing the distribution of principal and interest payments for the various periodic payments.
Example 22

A loan of RM1,000 at 12% compounded monthly is to be amortised by 18 monthly payments.

(a) Calculate the monthly payment.

(b) Construct an amortisation schedule.
Solution

(a) From $A = Ra \ n \ i$, we get

$1,000 = Ra_{18.1\%}$

$1,000 = R(16.398)$

$R = RM60.98$

Monthly payment is RM60.98
### (b) Amortisation schedule

<table>
<thead>
<tr>
<th>Period</th>
<th>Beginning balance (RM)</th>
<th>Ending balance (RM)</th>
<th>Monthly payment (RM)</th>
<th>Total principal paid (RM)</th>
<th>Interest paid (RM)</th>
<th>Total paid (RM)</th>
<th>Total balance (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,000.00</td>
<td>949.02</td>
<td>60.98</td>
<td>60.98</td>
<td>60.98</td>
<td>102.47</td>
<td>10.00</td>
</tr>
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<td>949.02</td>
<td>897.53</td>
<td>60.98</td>
<td>121.96</td>
<td>154.48</td>
<td>182.95</td>
<td>28.47</td>
</tr>
<tr>
<td>3</td>
<td>897.53</td>
<td>845.52</td>
<td>60.98</td>
<td>243.93</td>
<td>207.01</td>
<td>365.89</td>
<td>36.92</td>
</tr>
<tr>
<td>4</td>
<td>845.52</td>
<td>792.99</td>
<td>60.98</td>
<td>365.89</td>
<td>313.64</td>
<td>487.86</td>
<td>52.25</td>
</tr>
<tr>
<td>5</td>
<td>792.99</td>
<td>739.94</td>
<td>60.98</td>
<td>487.86</td>
<td>422.42</td>
<td>510.28</td>
<td>44.85</td>
</tr>
<tr>
<td>6</td>
<td>739.94</td>
<td>686.36</td>
<td>60.98</td>
<td>609.82</td>
<td>533.39</td>
<td>643.19</td>
<td>52.25</td>
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<td>7</td>
<td>686.36</td>
<td>632.24</td>
<td>60.98</td>
<td>732.24</td>
<td>655.11</td>
<td>777.35</td>
<td>64.85</td>
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<tr>
<td>8</td>
<td>632.24</td>
<td>577.58</td>
<td>60.98</td>
<td>854.58</td>
<td>677.51</td>
<td>932.09</td>
<td>76.44</td>
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<td>577.58</td>
<td>522.37</td>
<td>60.98</td>
<td>977.37</td>
<td>700.25</td>
<td>1,077.62</td>
<td>76.44</td>
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<tr>
<td>10</td>
<td>522.37</td>
<td>466.61</td>
<td>60.98</td>
<td>1,099.61</td>
<td>723.13</td>
<td>1,222.74</td>
<td>76.44</td>
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<tr>
<td>11</td>
<td>466.61</td>
<td>410.30</td>
<td>60.98</td>
<td>1,220.90</td>
<td>746.08</td>
<td>1,366.98</td>
<td>76.44</td>
</tr>
<tr>
<td>Period</td>
<td>Beginning balance (RM)</td>
<td>Ending balance (RM)</td>
<td>Monthly payment (RM)</td>
<td>Total paid (RM)</td>
<td>Total principal (RM)</td>
<td>Total interest (RM)</td>
<td>Total (RM)</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------</td>
<td>---------------------</td>
<td>----------------------</td>
<td>----------------</td>
<td>----------------------</td>
<td>---------------------</td>
<td>------------</td>
</tr>
<tr>
<td>12</td>
<td>410.30</td>
<td>353.42</td>
<td>60.98</td>
<td>731.78</td>
<td>646.58</td>
<td>85.20</td>
<td>1000.00</td>
</tr>
<tr>
<td>13</td>
<td>353.42</td>
<td>295.97</td>
<td>60.98</td>
<td>792.77</td>
<td>704.03</td>
<td>88.74</td>
<td>1097.69</td>
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<tr>
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<td>237.95</td>
<td>60.98</td>
<td>853.75</td>
<td>762.05</td>
<td>91.70</td>
<td>1164.70</td>
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<tr>
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<td>237.95</td>
<td>179.35</td>
<td>60.98</td>
<td>914.73</td>
<td>820.65</td>
<td>94.08</td>
<td>1205.48</td>
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<tr>
<td>16</td>
<td>179.35</td>
<td>120.16</td>
<td>60.98</td>
<td>975.71</td>
<td>879.84</td>
<td>95.87</td>
<td>1275.62</td>
</tr>
<tr>
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<td>120.16</td>
<td>60.38</td>
<td>60.98</td>
<td>1036.69</td>
<td>939.62</td>
<td>97.07</td>
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<td>60.38</td>
<td>0.00</td>
<td>60.98</td>
<td>1097.68</td>
<td>1000.00</td>
<td>97.68</td>
<td>1207.68</td>
</tr>
</tbody>
</table>
2.4.4 Sinking fund

When a loan is settled by the sinking fund method, the creditor will only receive the periodic interests due. The face value of the loan will only be settled at the end of the term. In order to pay this face value, the debtor will create a separate fund in which he will make periodic deposits over the term of the loan. The series of deposits made will amount to the original loan.
Example 23

A debt of RM1,000 bearing interest at 10% compounded annually is to be discharged by the sinking fund method. If five annual deposits are made into a fund which pays 8% compounded annually,

(a) find the annual interest payment,

(b) find the size of the annual deposit into the sinking fund,

(c) what is the annual cost of his debt,

(d) construct the sinking fund schedule.
Solution

(a) Annual interest payment = RM1,000 × 10%
   = RM100

(b) From $S = Rs_{n_i}$, we get
   
   $1,000 = Rs_{5.8\%}$
   
   $R = RM170.46$

(c) Annual cost = Annual interest payment + Annual deposit
   
   $= RM100 + RM170.46$
   
   $= RM270.46$
### End of Interest Annual Amount
<table>
<thead>
<tr>
<th>period</th>
<th>earned</th>
<th>deposit</th>
<th>at the end of period (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (year)</td>
<td>0 (RM)</td>
<td>170.46</td>
<td>170.46</td>
</tr>
<tr>
<td>2</td>
<td>13.64*</td>
<td>170.46</td>
<td>354.56**</td>
</tr>
<tr>
<td>3</td>
<td>28.36***</td>
<td>170.46</td>
<td>553.38****</td>
</tr>
<tr>
<td>4</td>
<td>44.27</td>
<td>170.46</td>
<td>768.11</td>
</tr>
<tr>
<td>5</td>
<td>61.45</td>
<td>170.46</td>
<td>1,000.02*****</td>
</tr>
</tbody>
</table>

* $170.46 \times 8\% = 13.64$

** $170.46 + 13.64 + 170.46 = 354.46$

*** $354.56 \times 8\% = 28.36$

**** $354.56 + 28.36 + 170.46 = 553.38$

***** Discrepancy due to rounding of decimal points
Annuity with continuous compounding

Future value of annuity with continuous compounding,

\[ S = R \left( \frac{e^{kt} - 1}{e^{k/p} - 1} \right) \]

Present value of annuity with continuous compounding,

\[ A = R \left( \frac{1 - e^{kt}}{e^{k/p} - 1} \right) \]

where

- \( S \) = future value of annuity
- \( A \) = present value of annuity
- \( R \) = periodic payment or deposit
- \( e \) = natural logarithm
- \( k \) = annual continuous compounding rate
- \( t \) = time in years
- \( p \) = number of payments in one year
Example 24

James wins an annuity that pays RM1,000 at the end of every six months for four years. If money is worth 10% per annum continuous compounding, what is

(a) the future value of this annuity at the end of four years,

(b) the present value of this annuity?
Solution

(a) \[ S = R \left[ \frac{e^{kt} - 1}{e^{k/p} - 1} \right] \]
\[ S = 1000 \left[ \frac{e^{10\% (4)} - 1}{e^{10\% /2} - 1} \right] \]
\[ = RM9,592.63 \]

(b) \[ A = R \left[ \frac{(1 - e^{kt})}{e^{k/p} - 1} \right] \]
\[ = 1000 \left[ \frac{(1 - e^{-10\% (4)t})}{e^{10\% /2} - 1} \right] \]
\[ = RM6,430.13 \]
SUMMARY

1. Future value of annuity is
   \[ S = R \left[ \frac{(1 + i)^n - 1}{i} \right] \]
   where \( R \) = periodic payment (of same value)
   \( i \) = interest rate per interest period
   \( n \) = number of interest periods (or payments)

2. Present value of annuity is
   \[ A = \frac{R\left(1 - (1 + i)^{-n}\right)}{i} \]

3. For forborne annuity with \( n \) equal payments, the amount at the end of \( N \) interest periods is
   \[ S = \left\{ R\left[(1 + i)^n - 1\right]/i \right\} (1 + i)^{N-n} \]

4. To solve for periodic payment,
   \[ R = \frac{Si}{[(1 + i)^n - 1]} \]
   if future value of annuity, \( S \), is known, or
   \[ R = \frac{Ai}{[1 - (1 + i)^{-n}]} \] if the present value of annuity, \( A \), is known.
5. The number of payments, n, or interest rate per interest period, i, can also be solved if future value of annuity, S, or present value of annuity, A, is known.

6. **Amortisation** of a debt refers to a sequence of equal payments at equal intervals of time to discharge the principal and interest of the debt.

7. An **amortisation schedule** is a table showing the distribution of principal and interest payments for the various periodic payments.

8. A **sinking fund** is a separate fund created by a debtor to deposit a series of periodic payments which amounts to the original loan. The creditor will receive periodic interests and the full loan will only be settled at the end of the loan term.
9. Future value of annuity with continuous compounding,

\[ S = R \left( \frac{e^{kt} - 1}{e^{k/p} - 1} \right) \]

Present value of annuity with continuous compounding,

\[ A = R \left[ \frac{1 - e^{kt}}{e^{k/p} - 1} \right] \]

where \( S \) = future value of annuity
\( A \) = present value of annuity
\( R \) = periodic payment or deposit
\( e \) = natural logarithm
\( k \) = annual continuous compounding rate
\( t \) = time in years
\( p \) = number of payments in one year
2.5 Depreciation

• Let us assume that you buy a car. Your car loses some value each time you drive until the car stops running and has no value.

• This loss in the value of your car is known as **depreciation**.

• The calculation of the depreciation of an asset is important to ensure accuracy in the calculation of the tax return.
LEARNING OBJECTIVES

• By the end of this chapter, you should be able to
  ✔ explain the meaning of depreciation,
  ✔ explain the difference in the various depreciation methods,
  ✔ compute annual depreciation, accumulated depreciation and book value, and
  ✔ construct a depreciation schedule.
2.5 Depreciation

• Depreciation is an accounting procedure for allocating the cost of capital assets, such as buildings, machinery tools and vehicles over their useful life.

• Depreciation expenses allow firms to recapture the amount of money needed for replacement of the assets and to recover the original investments.
Terms related to depreciation

• **Original cost**
  The original cost of an asset is the amount of money paid for an asset plus any sales taxes, delivery charges, installation charges and other costs incurred.

• **Salvage value**
  The salvage value (scrap value or trade-in value) is the value of an asset at the end of its useful life.

• **Useful life**
  The useful life of an asset is the life expectancy of the asset or the number of years the asset is expected to last.

• **Total depreciation (Depreciable value)**
  The total depreciation or the wearing value of an asset is the difference between cost and scrap value.
• **Annual depreciation**
  The annual depreciation is the amount of depreciation in a year.

• **Accumulated depreciation**
  The accumulated depreciation is the total depreciation to date.

• **Book value**
  The book value or carrying value of an asset is the value of the asset as shown in the accounting record. It is the difference between the original cost and the accumulated depreciation charged to that date.

  For example a car which was purchased for RM40,000 two years ago, will have a book value of RM34,000 if its accumulated depreciation for the two years is RM6,000.

• Three methods of depreciation are commonly used. These methods are
  • straight line method,
  • declining balance method,
  • sum-of-year digits method.
2.5.1 Straight line method

- By straight line method, the total amount of depreciation is spread evenly to each accounting period throughout the useful life of the asset.
- Annual depreciation = \( (\text{cost} - \text{salvage value}) / \text{useful life} \)
  \[ = \frac{\text{total depreciation}}{\text{useful life}} \]
- Annual rate of depreciation
  \[ = \left[ \frac{\text{annual depreciation}}{\text{total depreciation}} \right] \times 100\% \]
  or
- total depreciation = \( \frac{1}{\text{useful life}} \times 100\% \)
- Book value = cost – accumulated depreciation
Example 1

- Lau Company bought a lorry for RM38,000. The lorry is expected to last five years and its salvage value at the end of five years is RM8,000.
- Using the straight line method,
  - (a) calculate the annual depreciation,
  - (b) calculate the annual rate of depreciation,
  - (c) calculate the book value of the lorry at the end of the third year,
  - (d) prepare a depreciation schedule.
Solution

• (a) **Here** Cost = RM38,000, Salvage value = RM8,000
Total depreciation = RM38,000 – RM8,000 = RM30,000
Useful life = 5 years
Annual depreciation = (cost - salvage value) / useful life
   = (RM38,000 – RM8,000)/5
   = RM6,000
(b) Annual rate of depreciation = (6,000/30,000) × 100% = 20%
OR
Annual rate of depreciation = (1/useful life) × 100
   = (1/5) × 100% = 20%
• (c) Book value at the end of the third year
   = cost – accumulated depreciation
   = RM38,000 – (3 × RM6,000)
   = RM20,000
## Depreciation schedule

<table>
<thead>
<tr>
<th>End of year</th>
<th>Annual depreciation (RM)</th>
<th>Depreciation to date (RM)</th>
<th>Book value at end of year (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>38,000</td>
</tr>
<tr>
<td>1</td>
<td>6,000</td>
<td>6,000</td>
<td>32,000</td>
</tr>
<tr>
<td>2</td>
<td>6,000</td>
<td>12,000</td>
<td>26,000</td>
</tr>
<tr>
<td>3</td>
<td>6,000</td>
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<td>6,000</td>
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<tr>
<td>5</td>
<td>6,000</td>
<td>30,000</td>
<td>8,000</td>
</tr>
</tbody>
</table>
Example 2

• The book values of an asset after the third year and fifth year using the straight line method are RM7,000 and RM5,000 respectively. What is the annual depreciation of the asset?

• Solution
Decline in value from third year to fifth year = RM7,000 – RM5,000

= RM2,000

This decline occurs within 2 years. Hence, annual depreciation of the asset

= RM2,000 /2

= RM1,000
2.5.2 Declining balance method

- The **declining balance method** is an accelerated method in which higher depreciation charges are deducted in the early life of the asset.

- If the original cost of the asset is $C$ and the rate of depreciation is $r\%$, then the depreciated values (book values) of the asset are calculated as follows.
  \[
  BV = C(1 - r)^n
  \]
  where $BV =$ book value
  $C =$ cost of asset
  $r =$ rate of depreciation
  $n =$ number of years

- The annual rate of depreciation is given by
  \[
  r = 1 - \frac{n}{\sqrt[1-n]{S/C}}
  \]
  where $r =$ annual rate of depreciation
  $n =$ useful life in years

- The accumulated depreciation $D_a$ up to $n$ years is given by
  \[
  D_a = C - C(1 - r)^n
  \]
Example 3

The cost of a fishing boat is RM150,000. The declining balance method is used for computing depreciation. If the depreciation rate is 15%, compute the book value and accumulated depreciation of the boat at the end of five years.

• Solution

Here,  \( C = \text{RM150,000} \),  \( r = 15\% \),  \( n = 5 \) years

Book value at the end of 5 years = \( C(1 - r)^n \)

\[
= 150,000 (1 - 0.15)^5 \\
= \text{RM66,555.80}
\]

Accumulated depreciation = cost of asset – book value

\[
= \text{RM150,000} - \text{RM66,555.80} \\
= \text{RM83,444.20}
\]
Example 4

Given cost of asset = RM15,000, useful life = 4 years, scrap value = RM3,000

(a) find the annual rate of depreciation,
(b) construct the depreciation schedule,
using the declining balance method.

• Solution

(a) Here, $C = RM15,000$, $S = RM3,000$, $t = 4$ years

From $r = 1 - \frac{n}{n\sqrt{(S/C)}}$, we get

$r = 1 - \frac{n}{n\sqrt{(3,000/15,000)}}$

$= 33.13\%$
(b) Depreciation

for the first year = 33.13% × 15,000 = RM4,969.50
for the second year = 33.13% × 10,030.50 = RM3,323.10
for the third year = 33.13% × 6,707.40 = RM2,222.16
<table>
<thead>
<tr>
<th>Year</th>
<th>Annual depreciation (RM)</th>
<th>Accumulated depreciation (RM)</th>
<th>Book value (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15000.00</td>
</tr>
<tr>
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<td>12000.00</td>
<td>3000.00</td>
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</table>
2.5.3 Sum-of-year digits method

- The **sum-of-year digits method** is another accelerated method.
- In this method, the rate of depreciation is based on the sum of the digits representing the number of years of the asset’s useful life.
- For example, if the asset has a useful life of three years, the sum of digits is \( S = 1 + 2 + 3 = 6 \).
- For an asset with a useful life of five years, the sum of digits is \( S = 1 + 2 + 3 + 4 + 5 = 15 \).
- \( S \) can be calculated with the formula
  \[
  S = \frac{n(n + 1)}{2}
  \]
  where \( S \) = sum of years’ digits
  \( n \) = useful life
- The amount of depreciation in the first year is \( \frac{n}{S} \) of the depreciable value of the asset, the second is \( \frac{n - 1}{S} \), the third is \( \frac{n - 2}{S} \) of the depreciable value and so on.
Example 5

A machine is purchased for RM45,000. Its life expectancy is five years with a zero trade-in value. Prepare a depreciation schedule using the sum-of-year digits method.

- Solution
  
  Useful life, \( n = 5 \), Sum of years’ digits, \( S = 1 + 2 + 3 + 4 + 5 = 15 \)

  OR

  \[ S = \frac{n(n + 1)}{2} = \frac{5(5 + 1)}{2} = 15 \]

  Amount of depreciation for each year is calculated as follows.
### Annual depreciation

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual depreciation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5/15 \times 45,000 = RM15,000</td>
</tr>
<tr>
<td>2</td>
<td>4/15 \times 45,000 = RM12,000</td>
</tr>
<tr>
<td>3</td>
<td>3/15 \times 45,000 = RM9,000</td>
</tr>
<tr>
<td>4</td>
<td>2/15 \times 45,000 = RM6,000</td>
</tr>
<tr>
<td>5</td>
<td>1/15 \times 45,000 = RM3,000</td>
</tr>
</tbody>
</table>
The depreciation schedule is as follows.

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual depreciation (RM)</th>
<th>Accumulated depreciation (RM)</th>
<th>Book value at the end of year (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>45,000</td>
</tr>
<tr>
<td>1</td>
<td>15,000</td>
<td>15,000</td>
<td>30,000</td>
</tr>
<tr>
<td>2</td>
<td>12,000</td>
<td>27,000</td>
<td>18,000</td>
</tr>
<tr>
<td>3</td>
<td>9,000</td>
<td>36,000</td>
<td>9,000</td>
</tr>
<tr>
<td>4</td>
<td>6,000</td>
<td>42,000</td>
<td>3,000</td>
</tr>
<tr>
<td>5</td>
<td>3,000</td>
<td>45,000</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 6

A computer is purchased for RM3,600. It is estimated that its salvage value at the end of eight years will be RM600. Find the depreciation and the book value of the computer for the third year using the sum-of year digits method.
Solution

\[ S = \frac{n(n + 1)}{2} \]
\[ = \frac{8(8 + 1)}{2} = 36 \]

Depreciable value = original cost – salvage value
\[ = \text{RM3,600} - \text{RM600} \]
\[ = \text{RM3,000} \]

Depreciation for the third year = \( \frac{6}{36} \times 3,000 = \text{RM500} \)

Accumulated depreciation for the first three years
\[ = \left( \frac{8}{36} + \frac{7}{36} + \frac{6}{36} \right) \times 3,000 \]
\[ = \text{RM1,750} \]

Book value = \text{RM3,600} – \text{RM1,750}
\[ = \text{RM1,850} \]
1. **Depreciation** is a decline in the value of assets due to age, wear and tear or decreasing efficiency.

2. **Terms** important in depreciation are **original cost**, **salvage value**, useful life, total depreciation, annual depreciation, accumulated, depreciation and book value.

3. The three common **methods of** depreciation are straight line method, declining balance method and sum-of-year digits method.

4. **In the straight line method**, the total amount of depreciation is spread evenly to each year throughout the useful life of the asset.

5. **In the declining balance method and sum-of-year digits method,**
   higher depreciation charges are deducted in the early years of the asset’s life.

6. In the straight line method:
   - **annual depreciation** = \( \frac{\text{cost} - \text{salvage value}}{\text{useful life}} \)
   - **book value** = \( \text{cost} - \text{accumulated depreciation} \)
7. In the declining balance method:

book value, \( BV = C(1 - r)^n \)

annual rate of depreciation \( r = 1 - \sqrt[n]{S/C} \)

accumulated depreciation, \( D_a = C - C(1 - r)^n \)

8. In the sum-of-year digits method, sum of digits,

\[ S = \frac{n(n + 1)}{2} \]

and

Depreciation in the first year is \( \frac{n}{S} \) of the depreciable value, the second is \( \frac{n - 1}{S} \) and so on.
2.6 Instalment Purchases

- In an instalment purchase, a down payment is made, followed by a series of regular payments (usually monthly or weekly).
- There are many retailers and wholesalers who sell their products on an instalment basis. You can buy an electrical item by paying a number of weekly or monthly instalments.
- When you are an instalment buyer, you are actually paying more than the cash price.
- The difference is the interest you have to pay for the credit given to you by the seller (unpaid balance), plus insurance and finance charges.
LEARNING OBJECTIVES

• By the end of this chapter, you should be able to
  ✔ explain the meaning of instalment purchases;
  ✔ understand how the interest rate is charged on the original balance and the reducing balance of credit;
  ✔ compute the interest rate charged on the original balance of credit;
  ✔ compute the interest rate charged on the reducing balance of credit, and
  ✔ compute the outstanding balance and unearned interest of the lender under the Rule of 78.
2.6.1 Interest charge based on original balance

- Normally, an instalment purchase requires a down payment. The original balance is given by
  \[ \text{original balance} = \text{cash price} - \text{down payment} \]
- The total amount paid in an instalment plan by the buyer is called instalment price; that is,
  \[ \text{instalment price} = \text{cash price} + \text{total interest} \]
  OR
  \[ \text{instalment price} = \text{down payment} + \text{total monthly payment} \]
- The monthly payment can be calculated by dividing the sum of the original balance and interest by the number of instalments; that is,
  \[ \text{monthly payment} = \frac{(\text{original balance} + \text{total interest})}{(\text{number of payments})} \]
Example 1

Jenny bought a refrigerator listed at RM800 cash through an instalment plan. She paid RM100 as a downpayment. The balance was settled by making ten monthly instalments. If the interest rate charged was 8.5% per annum on the original balance, find

(a) the total interest charged,
(b) the instalment price,
(c) the monthly payment.
Solution

- Original balance = cash price – down payment = RM800 – RM100
  
  = RM700

(a) From I = Prt, we get
Total interest = 700 × 8.5% × 10/12 = RM49.58

(b) Instalment price = cash price + total interest
  = RM800 + RM49.58
  = RM849.58

(c) Monthly payment = (original balance + total interest) / (number of payments) = (RM700 + RM49.58) / 10
  = RM74.96
Example 2

Kim Ean bought an electric appliance through an instalment plan in which she paid RM200 downpayment. She had to make 12 monthly payments of RM120 each to settle the unpaid balance. If the dealer charged her an interest of 5% per annum on the original balance, find the cash price of the item.
Solution

Total monthly payment = 12 \times 120 = RM1,440

Instalment price = down payment + total monthly payment

= RM200 + RM1,440

= RM1,640

Let the cash price of the item be RM K. Hence,

total interest charged = instalment price – cash price

= 1,640 – K

From I = Prt, we get

1,640 – K = (K – 200) \times 5\% \times 12/12

1,640 – K = 0.05K – 10

K = RM1,571.43

Thus, the cash price is RM1,571.43.
2.6.2 Interest charge based on reducing balance

• interest rate charged on reducing balance is an annual rate which is applied only to the balance due at the time of each payment.
• There are several methods.
• The annual rate based on annuity method is the effective rate. Other rates that are based on reducing balances are not effective rates but may be close approximations of the effective rate.
• We shall discuss only two methods of reducing balance, namely
  • annuity method
  • Constant Ratio formula
Annuity method

• The **annuity method** is also called the **amortisation method**.

• **The Federal** government in providing housing loans uses this method to compute the monthly instalments.

• Many banks also use this method of calculation but with some variations. Usually, the banks first determine the annual payment using the annual rest (interest rate) and then divide the annual payment by 12 to arrive at the monthly instalment.

• If A is the amount of loan borrowed, i the interest rate per interest period and n the number of interest periods or the number of instalment repayments, then
A = R[1 – (1 + i)^{-n}]/i

where R is the instalment payment for each period.

• Solving for R, we get

   R = Ai / [1 – (1 + i)^{-n}]

Thus, if the amount of the loan A, interest rate i, and the number of instalments are known, the amount of the instalment payment can be calculated.

• The formula also allows us to find the amount of the original loan if the interest rate, instalment payment and the number of instalments are known.
Example 3

A washing machine is selling for RM2,000 cash. Through an instalment purchase, the buyer has to pay RM400 downpayment and ten monthly instalments. If the interest charged is 8% per annum on reducing balance, find

(a) the monthly payment,

(b) the total interest charged,

(c) the instalment price,

by using the annuity method.
(a) Balance outstanding = RM1,600
From \( R = \frac{A_i}{1 - (1 + i)^{-n}} \), we get
\[ R = 1600 \times \frac{0.08 /12}{1 - (1 + 0.08/12)^{-10}} \]
\[ R = RM165.93 \]
Hence, the monthly payment is RM165.93.

(b) Total interest charged = total monthly payment – outstanding balance
\[ = 165.93 \times 10 - 1,600 \]
\[ = RM59.30 \]

OR
Total interest charged = instalment price – cash price
\[ = (400 + 165.93 \times 10) - 2,000 \]
\[ = RM59.30 \]

(c) Instalment price = down payment + total monthly payment
\[ = RM400 + (165.93 \times 10) = RM2,059.30 \]

OR
Instalment price = cash price + total interest charged
\[ = RM2,000 + RM59.30 = RM2,059.30 \]
Example 4

Jasmin purchased some equipment from a wholesaler. The wholesaler offered her terms under which 12% of the purchase price will be added to the purchase price and the debt would be settled by 12 monthly payments. She can borrow from a finance company which charges 15% compounded monthly and repay the finance company by making 12 monthly payments. Which alternative should Jasmin choose?
Solution

Let the cash price be **RM P**.

- **From the dealer**
  
  Total payment = P + 0.12P = 1.12P
  
  Monthly payment = 1.12P / 12 = 0.09333P

- **From the finance company**
  
  From R = Ai / [1 – (1 + i)^-n], we get
  
  R = P(0.0125) / [1 – (1 + 0.0125)^-12]
  
  = 0.0903P

  The monthly payment is 0.0903P.

- Hence, Jasmin should borrow from the finance company to buy the equipment.
The Constant Ratio formula is frequently used to approximate the actual annual percentage rate, APR or effective rate.

The Constant Ratio formula is given by

\[ r = \frac{2MI}{B(n + 1)} \]

where

- \( r \) = annual interest rate
- \( M \) = 12 for monthly instalments and 52 for weekly instalments
- \( I \) = total interest charged for instalment plan
- \( B \) = original outstanding balance or principal of original debt
- \( n \) = total number of instalments

The Constant Ratio formula can also be used to calculate the total interest charged if the interest rate on the reducing balance is given, that is,

\[ I = B(n + 1)r/2M \]
Example 5

A washing machine is being sold for RM2,000 cash. Through an instalment purchase, the buyer has to pay RM400 downpayment and 10 monthly instalments. If the interest charged is 8% per annum on the reducing balance, find

(a) the total interest charged,
(b) the monthly payment,
(c) the instalment price,
by using the Constant Ratio formula.
(a) In using this method, we have to find the total interest $I$, charged first.

From $r = 2MI/B(n+1)$, we get

$$8\% = 2(12)I / [1,600(10 + 1)]$$

$$I = RM58.67$$

(b) Monthly payment = (original balance + total interest) / number of payments

$$= (RM1,600 + RM58.67) / 10$$

$$= RM165.87$$

(c) Instalment price = cash price + total interest charged

$$= RM2,000 + RM58.67$$

$$= RM2,058.67$$
Example 6

Zaleha purchased an RM8,000 piano through an instalment plan. She has to pay RM2,000 downpayment and 18 monthly payments of RM350 each. Find the

(a) instalment price,

(b) total interest charged,

(c) flat rate (simple interest rate) charged,

(d) approximate APR by using the Constant Ratio formula.
Solution

(a) Instalment price = down payment + total monthly payment
\[= 2,000 + 18(350)\]
\[= RM8,300\]

(b) Total interest = instalment price – cash price
\[= RM8,300 – RM8,000\]
\[= RM300\]

(c) From \(I = Prt\), we get
\[300 = 6,000 \times r \times 18/12\]
\[r = 3.33\%\]

(d) \[r = \frac{2MI}{B(n + 1)}\]
\[= \frac{2(12)(300)}{6,000(18 + 1)}\]
\[= 6.32\%\]
Example 7

Rosman purchased a RM4,000 computer. He has to pay RM2,000 downpayment and 20 weekly payments of RM110 each. Find the approximate effective rate that is charged by using the Constant Ratio formula.

• Solution

Balance outstanding = RM4,000 – RM2,000 = RM2,000
Total interest paid = 110 × 20 – 2,000 = RM200
From \( r = \frac{2MI}{B(n + 1)} \), we get
\[ r = \frac{2(52)(200)}{2,000(20 + 1)} \]
\[ = 49.52\% \]
Example 8

An electric guitar is priced at RM300 cash. A cash purchase is entitled to a 10% discount. The guitar can be purchased for RM50 downpayment and RM13 a week for 20 weeks. Find the approximate effective rate charged to the instalment buyer by using the Constant Ratio formula.
True cash price of guitar = 300(90%) = RM270
Instalment price = 50 + (13 \times 20) = RM310
Total interest charged = 310 – 270 = RM40
From \( r = \frac{2MI}{B(n + 1)} \), we get
\[
r = \frac{2(52)(40)}{250(20 + 1)}
\]
\[
= 79.24\%
\]
Example 9

Nelly bought a hi-fi set listed at RM800 cash through an instalment plan in which she had to make six monthly payments at 10% per annum simple interest. By using the Constant Ratio formula, what was the approximate effective rate charged by the dealer?

• Solution

From \( I = Prt \), we get

interest charged = \( 800 \times 10\% \times 0.5 = RM40 \)

Using the Constant Ratio formula, \( r = \frac{2MI}{B(n + 1)} \), we get

\[
r = \frac{2(12)(40)}{800(6 + 1)}
\]

\[
= 17.14\%
\]
2.6.3 Unequal instalment payments and repayment schedules

- Some instalment purchases may not require regular equal payments.
Example 10

Diana bought a RM4,600 stereo set on an instalment basis in which an interest of 1% per month on any outstanding balance was charged. She made a RM1,000 down payment. For the balance, she had to pay RM600 every month (principal payment) plus any interest due. Construct a repayment schedule to show the monthly payments.
Solution
Repayment schedule

<table>
<thead>
<tr>
<th>End of month</th>
<th>Principal payment (RM)</th>
<th>Interest charged (RM)</th>
<th>Monthly payment (RM)</th>
<th>Outstanding balance (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3,600</td>
</tr>
<tr>
<td>1</td>
<td>600</td>
<td>36 (3600x0.01x1)</td>
<td>636</td>
<td>3,000</td>
</tr>
<tr>
<td>2</td>
<td>600</td>
<td>30 (3000x0.01x1)</td>
<td>630</td>
<td>2,400</td>
</tr>
<tr>
<td>3</td>
<td>600</td>
<td>24 (2400x0.01x1)</td>
<td>624</td>
<td>1,800</td>
</tr>
<tr>
<td>4</td>
<td>600</td>
<td>18 (1800x0.01x1)</td>
<td>618</td>
<td>1,200</td>
</tr>
<tr>
<td>5</td>
<td>600</td>
<td>12 (1200x0.01x1)</td>
<td>612</td>
<td>600</td>
</tr>
<tr>
<td>6</td>
<td>600</td>
<td>6 (600x0.01x1)</td>
<td>606</td>
<td>-</td>
</tr>
</tbody>
</table>
Total interest charged = RM36 + RM30 + RM24 + RM18 + RM12 + RM6 = RM126

If equal payment is desired, then
monthly payment = (RM3,600 + RM126)/6

= RM621

It must be noted that this variation is identical to the Constant Ratio method.
Example 11

Rosita bought a television worth RM3,000 on an instalment basis in which she was charged 1% per month on any outstanding balance. She made a RM1,000 down payment and paid RM600 every month. Find the number of payments she made and the value of the final payment. Construct a repayment schedule.
Outstanding balance = RM2,000
Interest charged = 2,000 \times 1\% = RM25
Amount due = RM2,000 + RM25 = RM2,025
Unpaid balance after first payment = RM2,025 – RM600 = RM1,425
Interest charged = 1,425 \times 1\% = RM17.81
Amount due = RM1,425 + RM17.81 = RM1,442.81
Unpaid balance after second payment = RM1,442.81 – RM600 = RM842.81
Interest charged = 842.81 \times 1\% = RM10.54
Amount due = RM842.81 + RM10.54 = RM853.35
Unpaid balance after third payment = RM853.35 – RM600 = RM253.35
Interest charged = RM253.35 \times 1\% = RM3.17
Final fourth payment = RM253.35 + RM3.17 = RM256.52
Rosita had to make three payments of RM600 and a final fourth payment of RM256.52.
## Repayment schedule

<table>
<thead>
<tr>
<th>End of month</th>
<th>Amount owed (RM)</th>
<th>Periodic payment (RM)</th>
<th>Outstanding balance (RM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000.00</td>
<td>-</td>
<td>200.00</td>
</tr>
<tr>
<td>1</td>
<td>2025.00</td>
<td>600.00</td>
<td>1425.00</td>
</tr>
<tr>
<td>2</td>
<td>1442.81</td>
<td>600.00</td>
<td>842.81</td>
</tr>
<tr>
<td>3</td>
<td>853.35</td>
<td>600.00</td>
<td>253.35</td>
</tr>
<tr>
<td>4</td>
<td>253.35</td>
<td>256.52</td>
<td>0.00</td>
</tr>
</tbody>
</table>
In Malaysia, under the Hire-Purchase Act (1967) which controls the hire purchase agreement, the **Rule of 78** (which is sometimes called the sum of digits method) is used to calculate the outstanding balance of any hire purchase agreement.

When a hire purchase loan is paid by an instalment plan, the presumption is that the consumer borrows the finance charge (interest) as well as the principal. If the consumer decides to settle the balance of a debt early, then the lender is not entitled to all the finance charge in the hire purchase agreement.

The buyer is entitled to a rebate of the unearned finance charge.

The Rule of 78 states that the outstanding balance of a hire purchase loan with a flat (simple) interest rate is given by

\[ B = RN - I \left[ \frac{(1 + 2 + 3 + \ldots + N)}{(1 + 2 + 3 + \ldots + n)} \right] \] or

\[ B = RN - I \left[ \frac{N(N+1)}{n(n+1)} \right] \]

where
- \( R \) = monthly payment
- \( N \) = number of payments yet to be settled
- \( I \) = total interest charged
- \( n \) = total number of payments
The expression I [(1 + 2 + 3 + ... + N)/(1+2+3+…+n)] is called the **unearned interest**.

Many instalment purchases are made for a period of one year where the sum of 12 months (1 + 2 + 3 + ... + 10 + 11 + 12) is equal to 78. This sum can also be obtained by using the formula

\[ n \frac{n + 1}{2} = 12 \frac{12 + 1}{2} = 78. \]

**Other instalment periods are given below**

<table>
<thead>
<tr>
<th>Number of months</th>
<th>Sum of months’ digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>171</td>
</tr>
<tr>
<td>24</td>
<td>300</td>
</tr>
<tr>
<td>30</td>
<td>465</td>
</tr>
<tr>
<td>36</td>
<td>666</td>
</tr>
</tbody>
</table>
The finance charge on a one-year hire purchase loan is RM390. Find the interest that was unearned by the lender if the loan was settled two months early.

- **Solution**

Interest unearned by the lender

\[
= I \left[ \frac{1 + 2 + 3 + \ldots + N}{1 + 2 + 3 + \ldots + n} \right] \\
= \text{RM}390 \left[ \frac{1 + 2}{78} \right] \\
= \text{RM}15
\]
Example 13

A loan of RM10,000 at a flat rate of 10% per annum was repaid by making 24 monthly instalments. Find
(a) the total interest charged,
(b) the monthly payment,
(c) the outstanding loan just after the tenth payment using the Rule of 78.

Solution
(a) Total interest charged = Prt = 10,000 × 10% × 2 = RM2,000
(b) Monthly payment = loan + interest = 12,000/24 = RM500
(c) Using the Rule of 78,
outstanding balance = RN − I(1 + 2 + 3+ ... + N)/( 1 + 2 + 3 + ... + n)
= (500 × 14) − 2,000 [(1 + 2 + 3 + ... + 14)/( 1 + 2 + 3 + ... + 24)]
= RM6,300
Example 14

Michelle borrowed a certain sum of money under a hire purchase agreement from a finance company. She has to repay RM250 monthly for 24 months. How much did she borrow if the finance company charged a flat rate of 10% per annum?

• **Solution**

Let amount borrowed = P ringgit

Total payment = $250 \times 24 = RM6,000$

Interest charged = $6,000 - P$

From the formula $I = Prt$, we get

$6,000 - P = P \times 0.1 \times 2$

$P = RM5,000$
Example 15

Juliet purchased a car listed at RM49,000 from Car Finance Ltd. through a hire purchase agreement in which she had to pay RM10,000 down and 24 monthly instalments of RM2,000 each. After one year of payment defaults, the car was legally repossessed and sold for RM25,000. Find the amount of refund that she would receive from the company or the amount of money she would have to pay.

• Solution

List price = RM49,000

Total payment = 10,000 + (24 × 2,000) = RM58,000

Hence the total interest charged is RM58,000 – RM49,000 = RM9,000.

Balance outstanding after one year

= RN – I( 1 + 2 + 3 + ... + N)/( 1 + 2 + 3 + ... + n)
= (2,000 × 12) – 9,000 [(1 + 2 + 3 + ... + 12)/( 1 + 2 + 3 + ... + 24)]
= RM21,660
Solution

List price = RM49,000
Total payment = 10,000 + (24 \times 2,000) = RM58,000
Hence the total interest charged is RM58,000 – RM49,000 = RM9,000.

Balance outstanding after one year
= RN – I( 1 + 2 + 3 + ... + N)/( 1 + 2 + 3 + ... + n)
= (2,000 \times 12) – 9,000 [(1 + 2 + 3 + ... + 12)/( 1 + 2 + 3 + ... + 24)] = RM21,660

• Since the amount of the sale is more than the outstanding balance, she is entitled to a refund of RM25,000 – RM21,660 = RM3,340.
1. An instalment purchase is a purchase in which a down payment is made and the balance is settled by a series of regular payments.

2. In instalment purchases, interest can be charged on the original balance by using the simple interest rate, or on the reducing balance using either the annuity method or Constant Ratio formula.

3. The formula used in interest charged on original balance is $I = Prt$.

4. The formula used in annuity method is $A = R\left[1 - (1 + i)^{-n}\right]/i$

5. The formula used in Constant Ratio is $r = \frac{2MI}{B(n + 1)}$
   where $r = $ annual interest rate
   $M = 12$ for monthly instalments and $52$ for weekly instalments
   $I = $ total interest charged for instalment plan
   $B = $ original outstanding balance or principal of original debt
   $n = $ total number of instalments

6. Original balance = cash price – down payment

7. Instalment price = down payment + total monthly payment
   = cash price + total interest
8. Monthly payment = (original balance + total interest)/number of payments
9. The Rule of 78 is used to calculate the outstanding balance of any hire purchase agreement.
10. The formula used to find outstanding balance is
    \[ B = RN - I\left(\frac{1 + 2 + 3 + \ldots + N}{1 + 2 + 3 + \ldots + n}\right) \]
    or
    \[ B = RN - I\left[\frac{N(N+1)}{n(n+1)}\right]\]
    where \( R \) = monthly payment
    \( N \) = number of payments yet to be settled
    \( I \) = total interest charged
    \( n \) = total number of payments
11. The expression \( I\left[\frac{1+2+3+\ldots+N}{1+2+3+\ldots+N}\right] \) is called the unearned interest.